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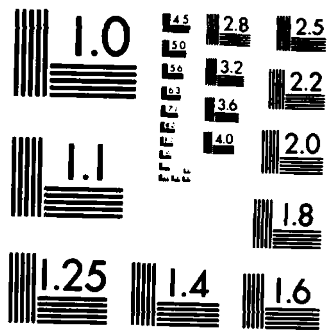

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THESIS ABSTRACT

NONLINEAR ATTITUDE STABILITY OF A DUAL-SPIN SPACECRAFT CONTAINING SPHERICAL DAMPERS

Paul Kenneth Winfree

Master of Science, June 8, 1984
B.A.E, Auburn University, 1983

69 Typed Pages

Directed by John E. Cochran, Jr.

A perturbation formulation and the generalized method of averaging are used to investigate the attitude motion of a symmetric dual-spin spacecraft which contains two spherical dampers arbitrarily located on the axisymmetric rotor and platform (one damper on each). The spherical dampers are intended to represent fully filled fuel tanks. The motions of these spherical dampers are treated as sources of perturbations which affect the otherwise torque-free attitude motion. Approximate attitude motion equations, valid through first order in the ratios of the damper moments of inertia to the spacecraft moment of inertia, are obtained. The generalized method of averaging is used to find an approximate analytical expression for the nutation angle for the case of equal damper moments of inertia and a constant speed rotor. This approximate expression is used to investigate the stability of the spacecraft attitude motion. Conditions for stability are presented and are found to agree well with a previous linear and energy-sink analyses obtained

for small nutation angles. Results obtained using the approximate analytical expression for the nutation angle are compared to corresponding numerical solution to the full set of nonlinear equations for several choices of system parameters and very good agreement is found, even for large nutation angles.

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NONLINEAR ATTITUDE STABILITY OF A DUAL-SPIN SPACECRAFT CONTAINING SPHERICAL DAMPERS

Paul Kenneth Winfree

A Thesis

Submitted to

the Graduate Faculty of

Auburn University

in Partial Fulfillment of the

Degree of

Master of Science

Auburn, Alabama

June 8, 1984

NONLINEAR ATTITUDE STABILITY OF A DUAL-SPIN SPACECRAFT

CONTAINING SPHERICAL DAMPERS

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NONLINEAR ATTITUDE STABILITY OF A DUAL-SPIN SPACECRAFT

CONTAINING SPHERICAL DAMPERS

Paul Kenneth Winfree

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VITA

Paul Kenneth Winfree, son of Paul Estas Winfree and Anne (Page) Winfree, was born October 25, 1950, in Lebanon, Tennessee. He attended public schools in Smith County, Tennessee, and graduated from Smith County High School in May 1968. He attended Cumberland College in Lebanon, Tennessee for one year, and then entered the United States Air Force in May 1970. In September, 1980, he entered Auburn University under a United States Air Force sponsored education and commissioning program. In March, 1983, he received a Bachelor of Aerospace Engineering degree from Auburn University. He then attended Officers Training School for three months and was commissioned a second lieutenant in the United States Air Force. He returned to graduate school at Auburn University in July 1983.

He married Diane Mims, daughter of Thomas Mims and June (Gwathney) Mann, in July 1977. He has one son, Steven Kenneth Winfree.

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LIST OF FIGURES (CONTINUED)

LIST OF SYMBOLS

English Symbols

[A]	Transformation matrix, inertial reference frame to platform-fixed reference frame.
a_{ij}	The ijth element of [A].
C_{P1}, C_{P2}, C_{P3}	Platform damping coefficients.
C_{R1}, C_{R2}, C_{R3}	Rotor damping coefficients.
CXYZ	Inertially fixed reference frame.
$Cx_1x_2x_3$	Platform-fixed reference frame.
$\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3$	Platform-fixed unit vector triad.
H_1, H_2, H_3	Platform-fixed components of the spacecraft rotational angular momentum.
h	Symmetry axis component of the rotor's angular momentum.
h_i	$h_{Pi} + h_{Ri}$, $i = 1,2,3$
h_{P1}, h_{P2}, h_{P3}	Platform-fixed components of the platform damper's angular momentum.
h_{R1}, h_{R2}, h_{R3}	Platform-fixed components of the rotor damper's angular momentum.
I_P	Moment of inertia of the platform damper.
I_R	Moment of inertia of the rotor damper.
I_S^*	$J_P + J_R + I_P + I_R$
I_t	Transverse moment of inertia of the spacecraft (excluding dampers).
I_t^*	$I_t + I_P + I_R$

LIST OF SYMBOLS (CONTINUED)

J_P	Spin axis moment of inertia of the platform (excluding dampers).
J_R	Spin axis moment of inertia of the rotor (excluding damper).
J_t	Transverse moment of inertia of the rotor (excluding damper).
k_{P1}, k_{P2}, k_{P3}	Inertial components of the platform damper's angular momentum.
k_{R1}, k_{R2}, k_{R3}	Inertial components of the rotor damper's angular momentum.
M_1, M_2, M_3	Components of external moment in Euler's moment equation.
T_{P1}, T_{P2}, T_{P3}	Torque components on the platform dampers.
T_{R1}, T_{R2}, T_{R3}	Torque components on the rotor dampers.
t	Time
x	$\cos\theta$
z	$\sin\theta$

Greek Symbols

α	$I_P/I_t = I_R/I_t$, small parameters.
δ_P	C_P/I_P
δ_R	C_R/I_R
ε	Small parameter.
Θ	Nutation angle.
Λ	Frequency parameter, $H_3(\frac{1}{J_P} - \frac{1}{I_t}) - h/J_P$
λ	Frequency parameter, H/I_t
μ_P	$\delta_P\lambda/(\delta_P^2+\lambda^2)$

μ_R	$\delta_R\lambda/(\delta_R^2+\lambda^2)$
ρ	J_P/I_t
σ	J_R/I_t
Φ	Angle of proper rotation.
$\dot{\phi}$	Angular speed of the rotor with respect to the platform.
Ψ	Precession angle.
Ω	Constant angular speed of the rotor with respect to the platform.
$\omega_1,\omega_2,\omega_3$	Platform-fixed components of the spacecraft angular velocity.
$\omega_{P1},\omega_{P2},\omega_{P3}$	Platform damper-fixed components of the angular velocity of the platform damper.
$\omega_{R1},\omega_{R2},\omega_{R3}$	Rotor damper-fixed components of the angular velocity of the rotor damper.

Other Nomenclature

[ ]	Indicates a square matrix.
{ }	Indicates a column matrix of vector components.
$(\dot{\ })$	Time rate of change of the elements of ( ).
$(\bar{\ })$	Averaged value of ( ).

I. INTRODUCTION

Spacecraft which contain spinning rotors and nominally despun platforms are called "dual-spin" spacecraft. In such a dual-spin configuration, the rotor provides gyroscopic stiffness for stability while the despun platform provides an oriented platform which usually contains scientific instruments, antennas, solar panels, and other components which must be oriented in a "fixed" direction.

A classical rigid-body analysis of the attitude stability of a spinning single body spacecraft indicates that a state of steady spin of the spacecraft is stable if rotation is about its principal axis of either minor or major moment of inertia. This classic stability criterion has been known to be inadequate since 1958 when the unanticipated instability of motion about the minor spin axis of the first U.S. satellite, Explorer I, was observed. The explanation of this instability is credited to R. N. Bracewell and O. K. Garriott, who modeled that spacecraft as a semirigid body which dissipated energy, and which therefore would approach the state of minimum energy which corresponded to rotation about its axis of maximum moment of inertia.[1]

In 1964, Landon and Stewart demonstrated that the motion of the spin axis of a symmetric dual-spin spacecraft with a despun platform may be stable if the spin axis is the axis of either maximum or minimum moments of inertia provided an energy dissipation device (or damper) is placed on the despun platform.[2] Independent development of the

dual-spin stabilization concept by A. J. Iorillo of Hughes Aircraft Company extended the scope of analysis to axisymmetric dual-spin vehicles with energy dissipating dampers on both the rotor and platform.[3] This extension allowed for the analysis of spacecraft of realistic complexity, since, in reality, internal motion on both the rotor and platform cause energy dissipation.

Although the problem of attitude stability of dual-spin spacecraft has been studied for more than two decades, it continues to be an area in which more can be learned. Several different methods can be used to study the attitude motion of a specific spacecraft, but if the type of configuration permits, a conventional linear stability analysis is a convenient first step. A linear stability analysis of a specific dual-spin spacecraft which contains arbitrarily located spherical dampers has been presented by Laskin, Sirlin, and Likins.[4] They also obtained corroborative results using an energy-sink method. In applying both methods, small nutation angles were assumed. An alternative approach to the energy-sink method is available, since when the energy dissipation is slow enough to justify the use of the energy-sink method, a perturbation method may be often used effectively. An analysis of this type was used by Cochran and Shu to analyze the nutational motion of a different spacecraft configuration and they obtained very good results.[5]

The model considered in Reference 4 is of the "ideal" type for analysis because it is axisymmetric and the energy dissipating devices on the rotor and platform are such that energy can be dissipated without changes in the inertia properties of the spacecraft. However, because the spacecraft attitude motion is nonlinear, the linear analysis

does not, and cannot be expected to, predict correctly the attitude stability of the spacecraft with respect to substantial perturbations in initial conditions. Laskin, et al., also found, via simulation on a digital computer, that in some cases, stability is dependent upon the ratio of the spacecraft's axial and transverse moments of inertia, a parameter which does not appear in the linear analysis equations.[4]

This thesis presents a "perturbation analysis" of the attitude motion of the type of spacecraft investigated in Reference 4. Following to some extent Cochran and Shu,[5] the motions of the spherical bodies are treated as perturbations of the attitude motion and the generalized method of averaging is used to find an approximate analytical expression for the nutation angle for the case of a constant-speed rotor. This expression approximates the nonlinear characteristics of the spacecraft motion and also contains the inertia ratio that is missing from the linear stability analysis. The expression obtained for the nutation angle is used to analyze the stability of the spacecraft's attitude motion.

The full nonlinear equations of motion are developed in Appendix A and numerical solutions of these equations and are used to verify the results obtained using the approximate solution.

The equations used in the generalized method of averaging are developed by treating the changes in the angular moment of the spherical dampers as perturbing torques. The platform-fixed components of the total angular momentum of the spacecraft are used as dependent variables along with the components of angular momenta of the dampers and the rotor's spin axis angular momentum. Euler angles which

define the attitude of the platform are then used to make a change of variables which transforms the angular momentum into equations of the "normal form" required to use the generalized method of averaging.[6,7] A brief description of the generalized method of averaging is given in Appendix B.

The generalized method of averaging is used to produce an expression that approximates the nutation angle of the spacecraft's attitude motion when the rotor is spinning at a constant speed with respect to the platform. This expression, which contains all the spacecraft's inertia and damping parameters, is used to determine regions of stable and unstable spacecraft attitude motion in the damping parameter plane for several spacecraft inertia parameters. Results obtained using the solution for the "averaged" nutation angle are also compared to those found by numerically integrating the full set of nonlinear equations for several "constant-speed" rotor cases.

II. SPACECRAFT MODEL

The type of spacecraft considered in this investigation is a highly symmetric dual-spin configuration such as that depicted in Fig. 2. The physical model of this type spacecraft consists of two axisymmetric rigid bodies, a nominally despun platform and a rapidly spinning rotor, connected by a shaft running along their collinear axes of symmetry. Both the platform and rotor contain arbitrarily located spherical dampers. These dampers may be thought of as fully loaded fuel tanks which are modeled as rigid spheres with constant surface damping coefficients.[4] Energy dissipation occurs when the spherical dampers rotate with respect to their respective containers and is due to the torques which oppose such motion of the dampers. The spherical nature of the dampers allows for energy dissipation with no change in the inertia properties of the spacecraft.

In Fig. 1, the $Cx_1x_2x_3$ coordinate system has its origin at the center of mass of the spacecraft and rotates with the platform. The unit vector triad $(\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3)$ is attached to $Cx_1x_2x_3$. The angular velocity of the rotor with respect to the platform is $\underline{\Omega} = \Omega\hat{\underline{e}}_3$.

In this analysis, the translational motion of the spacecraft is not considered and its attitude motion is assumed torque free.

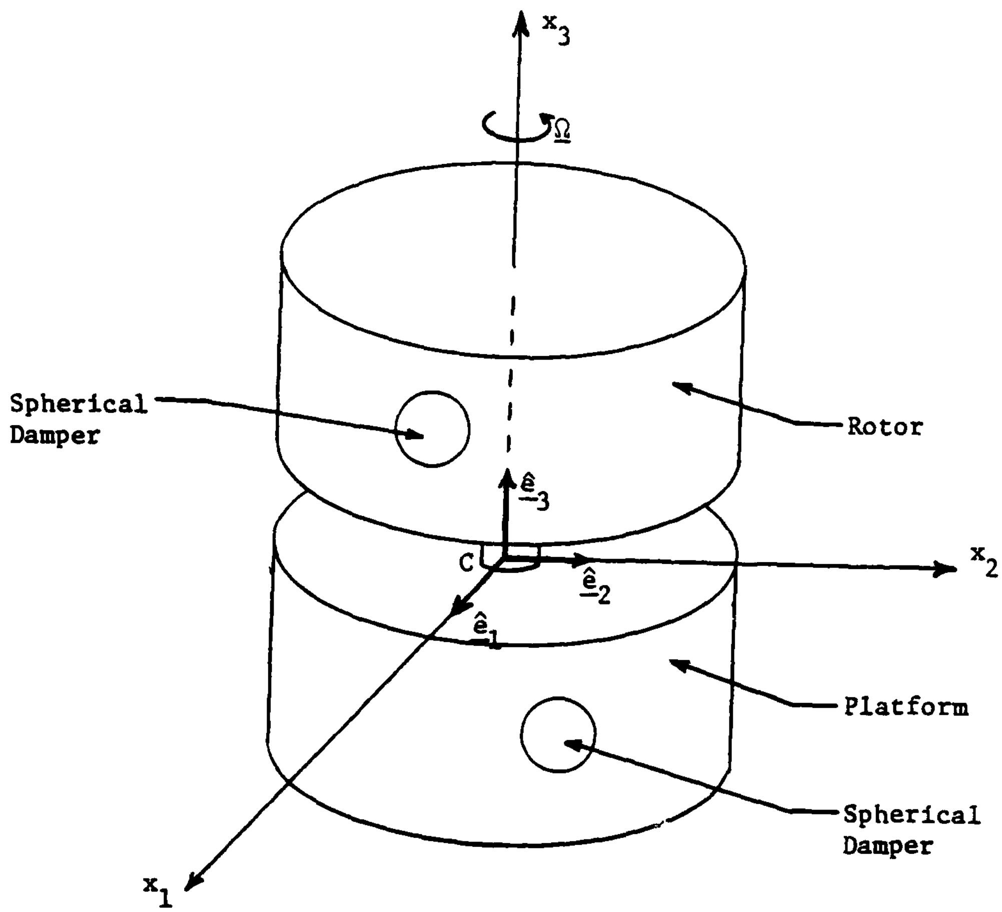

Figure 1. Spacecraft Configuration

III. EQUATIONS FOR USE IN THE GENERALIZED METHOD OF AVERAGING

Attitude Motion of the Spacecraft

The equations for the attitude motion of the spacecraft may be developed in terms of $\underline{H}$, the angular momentum of the system. The platform-fixed components of $\underline{H}$ are

$$H_1 = I_t\omega_1 + I_P(\omega_{P1}+\omega_1) + I_R(\omega_{R1}+\omega_1) \ , \tag{1a}$$

$$H_2 = I_t\omega_2 + I_P(\omega_{P2}+\omega_2) + I_R(\omega_{R2}+\omega_2) \tag{1b}$$

and

$$H_3 = J_P\omega_3 + J_R(\omega_3+\dot{\phi}) + I_P(\omega_{P3}+\omega_3) + I_R(\omega_{R3}+\omega_3+\dot{\phi}) \ . \tag{1c}$$

The angular momentum of the dampers has platform-fixed components

$$h_{Pi} = I_P(\omega_{Pi}+\omega_i) \quad , \text{ for } i = 1,2,3 \ , \tag{2a}$$

$$h_{Ri} = I_R(\omega_{Ri}+\omega_i) \quad , \text{ for } i = 1,2 \ , \tag{2b}$$

and

$$h_{R3} = I_R(\omega_{R3}+\omega_3+\dot{\phi}). \tag{2c}$$

Hence, Eqs. (1) may be rewritten as

$$H_1 = I_t\omega_1 + h_{P1} + h_{R1}, \tag{3a}$$

$$H_2 = I_t\omega_2 + h_{P2} + h_{R2} \tag{3b}$$

and

$$H_3 = J_P\omega_3 + h_{P3} + h_{R3} + h \ , \tag{3c}$$

where $h = J_R(\omega_3+\dot{\phi})$, is the symmetry axis component of the rotor's angular momentum.

If the no angular momentum compor ıts of the platform and rotor dampers are combined into single terms, i.e.,

$$\{h_i\} = \{h_{Pi}\} + \{h_{Ri}\} \quad , \text{ for } i = 1, 2, 3 \ , \tag{4}$$

then the following expressions for ω_1, ω_2, and ω_3 result from Eq. (3).

$$\omega_1 = (H_1-h_1)/I_t \ , \tag{5a}$$

$$\omega_2 = (H_2-h_2)I_t \tag{5b}$$

and

$$\omega_3 = (H_3-h-h_3)/J_P \tag{5c}$$

When Euler's moment equation is applied to $\underline{H}$, the following equation is obtained:

$$\underline{M} = \overset{\circ}{\underline{H}} + \underline{\omega}x\underline{H} \tag{6}$$

Here, $\overset{\circ}{\underline{H}}$ is the time rate of change of $\{H\}$ as seen in the $Cx_1x_2x_3$ system and $\underline{M}$ is the external torque about the spacecraft's center of mass. Since the motion of the spacecraft is considered torque free, the following matrix equation may be obtained from Eq. (6).

$$\{\dot{H}\} = -[\tilde{\omega}] \ \{H\} \ , \tag{7}$$

where $\{\dot{H}\} = [\dot{H}_1 \ \dot{H}_2 \ \dot{H}_3]^T$ and

$$[\tilde{\omega}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \ . \tag{8}$$

Equation (7) may be expanded after the approximations for ω_1, ω_2, and ω_3 are substituted to obtain

$$\dot{H}_1 = H_2[H_3(\frac{1}{J_p} - \frac{1}{I_t}) - h/J_p] - H_2h_3/J_p + H_3h_2/I_t \ , \tag{9a}$$

$$\dot{H}_2 = -H_1[H_3(\frac{1}{J_p} - \frac{1}{I_t}) - h/J_p] + H_1h_3/J_p - H_3h_1/I_t$$

and

$$\dot{H}_3 = H_2h_1/I_t - H_1h_2/I_t \ . \tag{9c}$$

Letting

$$\Lambda = H_3(\frac{1}{J_p} - \frac{1}{I_t}) - \frac{1}{J_p} \quad , \tag{10}$$

Eqs. (7) may be rewritten as

$$\dot{H}_1 = H_2\Lambda - H_2h_3/J_p + H_3h_2/I_t, \tag{11a}$$

$$\dot{H}_2 = -H_1\Lambda + H_1h_3/J_p - H_3h_1/I_t \tag{11b}$$

and

$$\dot{H}_3 = H_2h_1/I_t - H_1h_2/I_t. \tag{11c}$$

Now, refer to Fig. 2 which shows the fixed reference frame CXYZ with the constant angular momentum vector, $\underline{H}$, aligned with the Z axis. The spacecraft axis system $Cx_1x_2x_3$ is obtained from the CXYZ system by a sequence of rotations through the Euler angles Ψ, Θ, and Φ. It can be seen from Fig. 2 that the Euler angles can be used to write the components of $\underline{H}$ in the form

$$H_1 = H \sin\Theta \sin\Phi, \tag{12a}$$

$$H_2 = H \sin\Theta \cos\Phi \tag{12b}$$

and

$$H_3 = H \cos\Theta. \tag{12c}$$

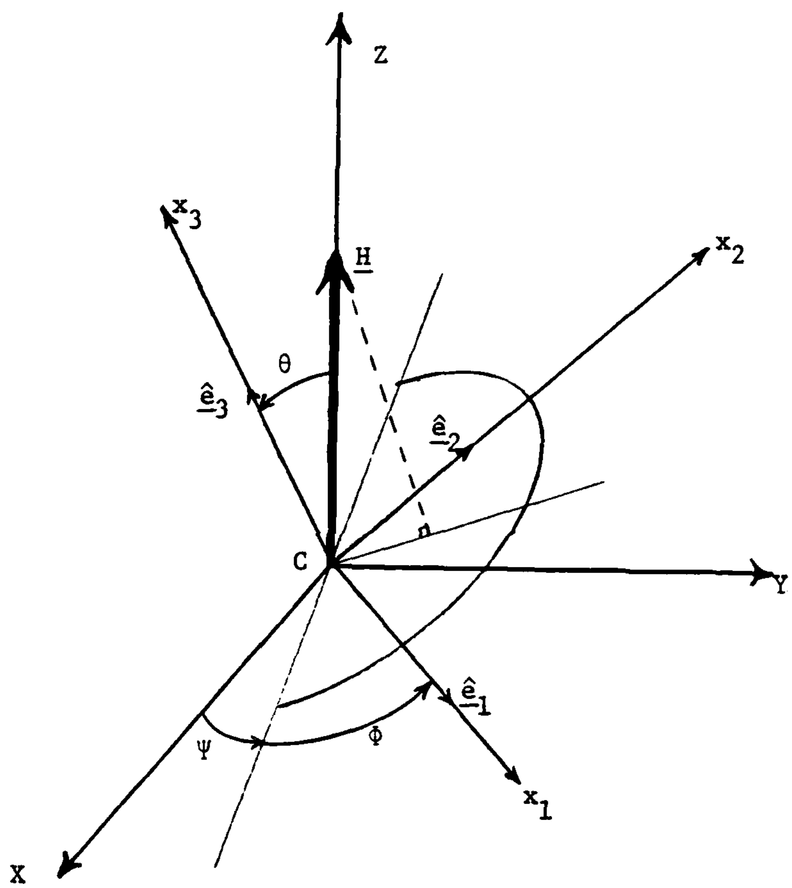

Figure 2. Euler Angles

By differentiating the above equations with respect to time, one finds that

$$\dot{H}_1 = H\dot{\Theta}\cos\Theta\,\sin\Phi + H\dot{\Phi}\sin\Theta\,\cos\Phi, \tag{13a}$$

$$\dot{H}_2 = H\dot{\Theta}\cos\Theta\,\cos\Phi - H\dot{\Phi}\sin\Theta\,\sin\Phi \tag{13b}$$

and

$$\dot{H}_3 = -H\dot{\Theta}\sin\Theta. \tag{13c}$$

Equations (11), (12) and (13) may be used to obtain the following equations:

$$\dot{\Theta}\cos\Theta\,\sin\Phi + \dot{\Phi}\sin\Theta\,\cos\Phi =$$
$$\Lambda\sin\Theta\,\cos\Phi - \frac{h_3}{J_p}\sin\Theta\,\cos\Phi + \frac{h_2}{I_t}\cos\Theta\,, \tag{14a}$$

$$\dot{\Theta}\cos\Theta\,\cos\Phi - \dot{\Phi}\sin\Theta\,\sin\Phi =$$
$$-\Lambda\sin\Theta\,\sin\Phi + \frac{h_1}{I_t}\cos\Theta\,\sin\Phi - \frac{h_2}{I_t}\cos\Theta \tag{14b}$$

and

$$\dot{\Theta}\sin\Theta = \frac{h_2}{I_t}\sin\Theta\,\sin\Phi - \frac{h_1}{I_t}\sin\Theta\,\cos\Phi. \tag{14c}$$

Equations (14b) and (14c) may then be solved for $\dot{\Theta}$ and $\dot{\Phi}$, respectively. Hence,

$$\dot{\Theta} = \frac{h_2}{I_t}\sin\Phi - \frac{h_1}{I_t}\cos\Phi \tag{15a}$$

and

$$\dot{\Phi} = \Lambda - \frac{h_3}{J_p} + (h_1\sin\Phi + h_2\cos\Phi)/(I_t\,\tan\Theta). \tag{15b}$$

These are two attitude equations that will be used later in applying the generalized method of averaging.

From Fig. 2, it can be shown that the kinematic equations for the Euler angles are:

$$\dot{\Theta} = \omega_1 \cos\Phi - \omega_2 \sin\Phi, \tag{16a}$$

$$\dot{\Phi} = \omega_3 - \dot{\Psi} \cos\Theta \tag{16b}$$

and

$$\dot{\Psi} = (\omega_1 \sin\Phi + \omega_2 \cos\Phi)/\sin\Theta. \tag{16c}$$

At this point it is convenient to find approximations for ω_1, ω_2, and ω_3. The approximations are found by assuming that each of the damper inertias is much smaller than any of the spacecraft inertias. By invoking this assumption, approximations for ω_1, ω_2, and ω_3 may be obtained from Eqs. (5). These approximations are

$$\omega_1 \simeq \frac{H_1}{I_t}, \tag{17a}$$

$$\omega_2 \simeq \frac{H_2}{I_t} \tag{17b}$$

and

$$\omega_3 \simeq (H_3 - h)/J_p. \tag{17c}$$

If the above approximations are used in Eqs (16), the following equations for $\dot{\Theta}$, $\dot{\Phi}$, and $\dot{\Psi}$ may be found:

$$\dot{\Theta} = \frac{H_1}{I_t} \cos\Phi - \frac{H_2}{I_t} \sin\Phi + \mathcal{O}(\alpha), \tag{18a}$$

$$\dot{\Phi} = (H_3 - h)/J_p - \dot{\Psi} \cos\Theta + \mathcal{O}(\alpha) \tag{18b}$$

and

$$\dot{\Psi} = \left(\frac{H_1}{I_t}\sin\Phi + \frac{H_2}{I_t}\cos\Phi\right)/\sin\Theta + \mathcal{O}(\alpha) \tag{18c}$$

By substituting values for H_1, H_2, and H_3 from Eqs. (12), one gets the attitude equations

$$\dot{\Theta} = 0 \ + \mathcal{O}(\alpha), \tag{19a}$$

$$\dot{\Phi} = \Lambda \ + \mathcal{O}(\alpha) \tag{19b}$$

and

$$\dot{\Psi} = \frac{H}{I_t} + \mathcal{O}(\alpha). \tag{19c}$$

In the application of the generalized method of averaging the angles Φ and Ψ are referred to as "fast" variables because the zeroth-order approximations to their time rates of change are non-zero. The angle Θ is called a "slow" variable, because to zeroth-order it is constant.[7] If the terms of (α) are disregarded and $\lambda = \frac{H}{I_t}$, then the zeroth-order approximation for the Euler angles are

$$\Theta = \Theta_o, \tag{20a}$$

$$\Phi = \Lambda t + \Phi_o \tag{20b}$$

and

$$\Psi = \lambda t + \Psi_o. \tag{20c}$$

These approximations will be used later in the development of the zeroth-order approximations for the attitude motion of the dampers.

Attitude Motion of the Platform Damper

To develop the desired equations for the platform damper attitude motion in the desired form, it is necessary to use the inertial components of angular momentum. If $\{k_p\}$ is used to denote the matrix of inertial components of the angular momentum vector of the

platform damper and a 3-1-3 rotation through the angles Ψ, Θ, and Φ is used to transform from inertial components to platform-fixed components, then

$$\{h_p\} = [A]\ \{k_p\} \tag{21}$$

where $[A]$ is the transformation matrix,

$$[A] = \begin{bmatrix} \cos\Phi & \sin\Phi & 0 \\ -\sin\Phi & \cos\Phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Theta & \sin\Theta \\ 0 & -\sin\Theta & \cos\Theta \end{bmatrix} \begin{bmatrix} \cos\Psi & \sin\Psi & 0 \\ -\sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{22}$$

In its expanded form the $[A]$ matrix is

$$[A] = \begin{bmatrix} \cos\Phi\ \cos\Psi - \sin\Phi\ \cos\Theta\ \sin\Psi & \cos\Phi\ \sin\Psi + \sin\Phi\ \cos\Theta\ \cos\Psi & \sin\Phi\ \sin\Theta \\ -\sin\Phi\ \cos\Psi - \cos\Phi\ \cos\Theta\ \sin\Psi & -\sin\Phi\ \sin\Psi + \cos\Phi\ \cos\Theta\ \cos\Psi & \cos\Phi\ \sin\Theta \\ \sin\Theta\ \sin\Psi & -\sin\Theta\ \cos\Psi & \cos\Theta \end{bmatrix} \tag{23}$$

Euler's moment equation may be applied using the angular momentum of the platform damper to obtain the vector equation,

$$\dot{\underline{h}}_p + \underline{\omega} \mathrm{x}\ \underline{h}_p = \underline{T}_p = -\ C_p \underline{\omega}_p\ . \tag{24}$$

By writing Eq. (24) in matrix form and substituting $\{\omega_p\} = \frac{1}{I_p}\{h_p\} - \{\omega\}$, one may obtain the results,

$$\{\dot{h}_p\} = -\frac{C_p}{I_p}\{h_p\} + C_p\{\omega\} - [\tilde{\omega}]\ \{h_p\}. \tag{25}$$

If Eq. (21) is now differentiated with respect to time, a second equation for $\{\dot{h}_p\}$ is found to be

$$\{\dot{h}_p\} = [\dot{A}]\ \{k_p\} + [A]\ \{\dot{k}_p\}. \qquad (26)$$

However, it can be shown[8] that

$$[\dot{A}] = -[\tilde{\omega}]\ [A]. \qquad (27)$$

Hence, Eq. (26) may be written in the form,

$$\{\dot{h}_p\} = -[\tilde{\omega}]\ [A]\{k_p\} + [A]\ \{\dot{k}_p\}. \qquad (28)$$

Now, by equating Eq. (28) and Eq. (25) and using Eq. (21) for $\{h_p\}$, one may obtain an expression for $\{\dot{k}_p\}$; i.e.,

$$\{\dot{k}_p\} = -\frac{C_p}{I_p}\{k_p\} + C_p[A]^T\{\omega\}. \qquad (29)$$

A solution to Eq. (29) may be assumed in the form,

$$\{k_p\} = \{\alpha_p\}\ e^{-C_p t/I_p}. \qquad (30)$$

If Eq. (30) is differentiated with respect to time, a second form for $\{\dot{k}_p\}$, viz.,

$$\{\dot{k}_p\} = \{\dot{\alpha}_p\}e^{-C_p t/I_p} - \frac{C_p}{I_t}\{k_p\}. \qquad (31)$$

Now, by equating Eq. (29) and Eq. (31), one finds that

$$\{\dot{\alpha}_p\} = C_p e^{C_p t/I_p}[A]^T\{\omega\}. \qquad (32)$$

Equation (32) may be expanded to obtain

$$\dot{\alpha}_{P1} = C_p e^{C_p t/I_p}(a_{11}\omega_1 + a_{21}\omega_2 + a_{31}\ \omega_3), \qquad (33a)$$

$$\dot{\alpha}_{P2} = C_p e^{C_p t/I_p}(a_{12}\omega_1 + a_{22}\omega_2 + a_{32}\ \omega_3) \qquad (33b)$$

and

$$\dot{\alpha}_{P3} = C_P e^{C_P t/I_P}(a_{13}\omega_1 + a_{23}\omega_2 + a_{33}\,\omega_3) \tag{33c}$$

where a_{ij} are the elements of the [A] matrix.

If the approximations for Θ, Φ, and Ψ given in Eqs. (20) and the approximations for ω_1, ω_2, and ω_3 given by Eqs. (17) are used in Eqs. (33), it can be shown, after some algebra, that

$$\dot{\alpha}_{P1} = C_P\;\Lambda\;z\;e^{C_P t/I_P}\sin\Psi\;, \tag{34a}$$

$$\dot{\alpha}_{P2} = -C_P\;\Lambda\;z\;e^{C_P t/I_P}\cos\Psi \tag{34b}$$

and

$$\dot{\alpha}_{P3} = C_P[\lambda z^2 + x(H_3 - h)/J_P]e^{C_P t/I_P}, \tag{34c}$$

where $z = \sin\Theta$ and $x = \cos\Theta$.

The solutions to the above equations, integrated from $t_0=0$ to $t_1=t$, are

$$\alpha_{P1} = \frac{C_P \Lambda z}{(\frac{C_P}{I_P})^2 + \lambda^2}\left\{e^{C_P t/I_P}\left(\frac{C_P}{I_P}\sin\Psi - \lambda\cos\Psi\right) + \lambda\right\} + \alpha_{P10}, \tag{35a}$$

$$\alpha_{P2} = \frac{-C_P \Lambda z}{(\frac{C_P}{I_P})^2 + \lambda^2}\left\{e^{C_P t/I_P}\left(\frac{C_P}{I_P}\cos\Psi + \lambda\sin\Psi\right) - \frac{C_P}{I_P}\right\} + \alpha_{P20} \tag{35b}$$

and

$$\alpha_{P3} = I_P\{\lambda z^2 + xH_3 - h)/J_P\}\left(e^{C_P t/I_P} - 1\right) + \alpha_{P30}. \tag{35c}$$

When used in Eq. (30), the above expressions give the inertial components of the platform angular momentum as functions of time and the fast variable $\Psi=\lambda t$. These components are

$$k_{P1} = \frac{C_P \Lambda z}{(\frac{C_P}{I_P})^2 + \lambda^2} \left(\frac{C_P}{I_P} \sin\Psi - \lambda\cos\Psi + \lambda e^{-C_P t/I_P} \right) + \alpha_{P10} e^{-C_P t/I_P}, \quad (36a)$$

$$k_{P2} = \frac{-C_P \Lambda z}{(\frac{C_P}{I_P})^2 + \lambda^2} \left(\frac{C_P}{I_P} \cos\Psi + \lambda\sin\Psi - \frac{C_P}{I_P} e^{-C_P t/I_P} \right) + \alpha_{P20} e^{-C_P t/I_P} \quad (36b)$$

and

$$k_{P3} = I_P[\lambda z^2 + x(H_3 - h)/J_P](1 - e^{-C_P t/I_P}) + \alpha_{P30} e^{-C_P t/I_P}. \quad (36c)$$

Armed with an expression for the inertial components of angular momentum, one may find the relative components from Eq. (21), which in component form are

$$h_{P1} = a_{11}k_{P1} + a_{12}k_{P2} + a_{13}k_{P3}, \quad (37A)$$

$$h_{P2} = a_{21}k_{P1} + a_{22}k_{P2} + a_{23}k_{P3} \quad (37b)$$

and

$$h_{P3} = a_{31}k_{P1} + a_{32}k_{p2} + a_{33}k_{P3} . \quad (37c)$$

If Eqs. (37) are expanded using the approximations for Θ, Ψ, and Φ in the transformation matrix and the transient terms in $\{k_P\}$ are neglected, then it can be shown, again with some algebra, that:

$$h_{P1} = \frac{C_P z\Lambda}{(C_P/I_P)^2 + \lambda^2} [-\lambda\cos\Phi - \frac{C_P}{I_P} x \sin\Phi]$$

$$+ I_P z \sin\Phi \; [\lambda z^2 + x(H_3 - h)/J_P] , \quad (38a)$$

$$h_{P2} = \frac{C_P z\Lambda}{(C_P/I_P)^2 + \lambda^2} [\lambda\sin\Phi - \frac{C_P}{I_P} x \cos\Phi]$$

$$+ I_P \; z\cos\Phi \; [\lambda z^2 + x(H_3 - h)/J_P] \quad (38b)$$

and

$$h_{P3} = \frac{C_P^2 z^2 \Lambda / I_P}{(C_P/I_P)^2 + \lambda^2} + I_P\ x[\lambda z^2 + x(H_3 - h)/J_P]. \qquad (38c)$$

The above expressions give the angular momentum components of the platform dampers as functions of the slow variable Θ found in x and z and the fast variables Φ and Ψ.

Motion of the Rotor Dampers

The equations for the rotor damper motion may be developed in the same manner as those for the platform. If $\{k_R\}$ denotes the inertial angular momentum, then the same transformation exists for the rotor dampers, i.e.,

$$\{h_R\} = [A]\ \{k_R\}\ . \qquad (39)$$

If a vector $\{\dot{\phi}\}$ is defined such that

$$\{\dot{\phi}\} = (0,\ 0,\ \dot{\phi})^T\ , \qquad (40)$$

then Eqs. (2b) and (2c) can be written in matrix form as

$$\{h_R\} = I_R\{\omega_R\} + I_R\{\omega\} + I_R\{\dot{\phi}\}. \qquad (41)$$

Euler's moment equation applied to the angular momentum of the rotor gives

$$\dot{\underline{h}}_R + \underline{\omega} x \underline{h}_R = \underline{T}_R\ . \qquad (42)$$

Switching to the matrix notation and substituting $\{\omega_R\} = \frac{1}{I_R}\{h_R\} - \{\omega\} - \{\dot{\phi}\}$, into Eq. (42) yield

$$\{\dot{h}_R\} = -C_R/I_R\ \{h_R\} + C_R\{\omega\} + C_R\{\dot{\phi}\} - [\tilde{\omega}]\ \{h_R\}\ . \qquad (43)$$

By differentiating Eq. (39) with respect to time,one obtains

$$\{\dot{h}_R\} = [\dot{A}]\ \{k_R\} + [A]\ \{\dot{k}_R\} = -\ [\tilde{\omega}]\ [A]\ \{k_R\} + [A]\ \{\dot{k}_R\} \tag{44}$$

Now, by equating Eqs. (43) and (44), it follows that

$$\{\dot{k}_R\} = -\ C_R/I_R\ \{k_R\} + C_R[A]^T\ \{\omega\} + C_R[A]^T\ \{\dot{\phi}\}. \tag{45}$$

As with the platform dampers, the solution for $\{k_R\}$ is assumed to be

$$\{k_R\} = \{\alpha_R\}e^{-C_Rt/I_R}. \tag{46}$$

By differentiating Eq. (46) with respect to time and equating the result to Eq. (45), one may obtain the following expression for $\{\dot{\alpha}_R\}$:

$$\{\dot{\alpha}_R\} = C_Re^{C_Rt/I_R}[A]^T\ \{\omega\} + C_Re^{C_Rt/I_R}[A]^T\ \{\dot{\phi}\}\ . \tag{47}$$

If the approximate expression for $\{\omega\}$ and the approximations to Θ, Φ, and Ψ are used in the transformation matrix in Eq. (47) then the results for the $\dot{\alpha}_{Rj}$ are

$$\dot{\alpha}_{R1} = C_Rz(\Lambda+\dot{\phi})e^{C_Rt/I_R}\sin\Psi, \tag{48a}$$

$$\dot{\alpha}_{R_2} = -C_Rz(\Lambda+\dot{\phi})e^{C_Rt/I_R}\ \cos\Psi \tag{48b}$$

and

$$\dot{\alpha}_{R3} = C_R[\lambda z^2 + x(H_3-h)/J_p + x\ \dot{\phi}]e^{C_Rt/I_R}\ . \tag{48c}$$

Equations (48) may be integrated from $t_0=0$ to $t_1=t$ to get the following results:

$$\alpha_{R_1} = \frac{C_Rz(\Lambda+\dot{\phi})}{(C_R/I_R)^2 + \lambda^2}\left\{e^{C_Rt/I_R}\left(\frac{C_R}{I_R}\sin\Psi - \lambda\cos\Psi\right) + \lambda\right\} + \alpha_{R10}, \tag{49a}$$

$$\alpha_{R2} = \frac{-C_Rz(\Lambda+\dot{\phi})}{(C_R/I_R)^2 + \lambda^2}\left\{e^{C_Rt/I_R}\left(\frac{C_R}{I_R}\cos\Psi + \lambda\sin\Psi\right) - \frac{C_R}{I_R}\right\} + \alpha_{R20} \tag{49b}$$

and

$$\alpha_{R3} = I_R \, [\lambda z^2 + x\,(H_3-h)J_p + x\,\dot{\phi}](e^{C_R t/I_R}-1) + \alpha_{R30} \tag{49c}$$

When used in Eq. (46), Eqs. (49) provide the inertial components of the angular momentum of the rotor damper. These are

$$k_{R1} = \frac{C_R z(\Lambda+\dot{\phi})}{(C_R/I_R)^2 + \lambda^2}\,[\frac{C_R}{I_R}\sin\Psi - \lambda\cos\Psi + \lambda e^{-C_R t/I_R}] + \alpha_{R10}e^{-C_R t/I_R}\ , \tag{50a}$$

and

$$k_{R2} = \frac{-C_R z(\Lambda+\dot{\phi})}{(C_R/I_R)^2 + \lambda^2}\,[\frac{C_R}{I_R}\cos\Psi + \lambda\sin\Psi - \frac{C_R}{I_R}e^{-C_R t/I_R}] + \alpha_{R20}e^{-C_{R20}} \tag{50b}$$

$$k_{R3} = I_R\,[\lambda z^2 + x(H_3-h)/J_p + x\dot{\phi}](1-e^{-C_R t/I_R}) + \alpha_{R30}e^{-C_R t/I_R} \tag{50c}$$

From these expressions for the inertial angular momentum components, one may find the platform-fixed components of the angular momentum by using Eq. (39) with the zeroth-order approximation to the transformation matrix and $\{\omega\}$. The results are

$$h_{R1} = \frac{-C_A z(\Lambda+\dot{\phi})}{(C_R/I_R)^2 + \lambda^2}\,[\frac{C_R}{I_R}x\sin\Phi + \lambda\cos\Phi]$$

$$+ I_R\, z\sin\Lambda t[\lambda z^2 + x\,(H_3-h)/J_p + x\,\phi]\ , \tag{51a}$$

$$h_{R2} = \frac{C_R z(\Lambda+\dot{\phi})}{(C_R/I_R)^2 + \lambda^2}\,[\lambda\sin\Phi - \frac{C_R}{I_R}x\cos\Phi]$$

$$+ I_R\, z\cos\Lambda t[\lambda z^2 + x(H_p-h)/J_p + x\,\phi] \tag{51b}$$

and

$$h_{R3} = \frac{C_R^2 z^2(\Lambda+\dot{\phi})/I_R}{(C_R/I_R)^2 + \lambda^2} + I_R x[\lambda z^2 + x((H_3-h)/\dot{J}_p + x\,\phi] \tag{51c}$$

Equations (51) are zeroth-order approximations to components of the angular momentum of the rotor damper as functions of the slow variable Θ and the fast variables Φ and Ψ.

Transformation of the Equations to Normal Form

To apply the generalized method of averaging, it is necessary to have equations of the form

$$\{\dot{x}\} = \varepsilon\{X_1\} + \varepsilon^2\{X_2\} + \ldots \tag{52a}$$

$$\{\dot{y}\} = \{Y_o\} + \varepsilon\{Y_1\} + \varepsilon^2\{Y_2\} + \ldots \tag{52b}$$

where ε is a small constant, $\{x\}$ is a vector of slow variables, $\{y\}$ is a vector of fast variables, and the vector functions $\{X_i\}$ and $\{Y_j\}$ are periodic functions in the elements of $\{y\}$ with a period of 2π.[7]

To arrive at the desired normal form, one must combine the angular momentum solutions to get

$$\begin{aligned} h_1 = &- \lambda \cos\Phi \left[\frac{C_P z\Lambda}{(C_P/I_P)^2 + \lambda^2} + \frac{C_R z(\Lambda+\dot{\phi})}{(C_R/I_R)^2 + \lambda^2}\right] \\ &- x \sin\Phi \left[\frac{C_P^2 z\Lambda/I_P}{(C_P/I_P)^2 + \lambda^2} + \frac{C_R^2 z(\Lambda+\dot{\phi})/I_R}{(C_R/I_R)^2 + \lambda^2}\right] \\ &+ z \sin\Phi \,[\lambda z^2 + x(H_3-h)/J_P](I_P+I_R) + I_R x z\dot{\phi} \sin\Phi , \end{aligned} \tag{53a}$$

$$\begin{aligned} h_2 = &\lambda \sin\Phi \left[\frac{C_P z\Lambda}{(C_P/I_P)^2 + \lambda^2} + \frac{C_R z(\Lambda+\dot{\phi})}{(C_R/I_R)^2 + \lambda^2}\right] \\ &- x \cos\Phi \left[\frac{C_P^2 z\Lambda/I_P}{(C_P/I_P)^2 + \lambda^2} + \frac{C_R^2 z(\Lambda+\dot{\phi})/I_R}{(C_R/I_R)^2 + \lambda^2}\right] \\ &+ z \cos\Phi \,[\lambda z^2 + x(H_3-h)/J_P](I_P+I_R) + I_R x z\dot{\phi} \cos\Phi \end{aligned} \tag{53b}$$

and

$$h_3 = z\left[\frac{C_P^2 z\Lambda/I_P}{(C_P/I_P)^2 + \lambda^2} + \frac{C_R^2 z(\Lambda+\dot{\phi})/I_R}{(C_R/I_R)^2 + \lambda^2}\right]$$

$$+ x[\lambda z^2 + X(H_3-h)/J_P](I_P + I_R) + I_R x^2\dot{\phi} \quad . \tag{53c}$$

The above equations for h_1 and h_2 along with the approximation for Φ given by Eq. (15a) may be used to obtain the result,

$$\dot{\Theta} = \frac{\lambda}{I_t}\left[\frac{C_P z\Lambda}{(C_P/I_P)^2 + \lambda^2} + \frac{C_R z(\Lambda+\dot{\phi})}{(C_R/I_R)^2 + \lambda^2}\right] \tag{54}$$

Now, it is assumed that the inertias of the dampers are equal and $\alpha = I_P/I_t = I_R/I_t$ is adopted as the small parameter. Since

$$C_P/I_t = (I_P/I_t)(C_P/I_P) = \alpha\, C_P/I_P \tag{55}$$

and

$$C_R/I_t = (I_R/I_t)(C_R/I_R) = \alpha\, C_R/I_R, \tag{56}$$

Eq. (54) may be written in the form

$$\dot{\Theta} = \alpha\left[\frac{C_P\lambda z\Lambda/I_P}{(C_P/I_P)^2 + \lambda^2} + \frac{C_R\lambda z(\Lambda+\dot{\phi})/I_R}{(C_R/I_R)^2 + \lambda^2}\right] \quad . \tag{57}$$

Now, because z and Λ in the above equations are functions of x, it is chosen as the slow variable. From $x = \cos\Theta$ it follows that

$$\dot{x} = -\dot{\Theta}\sin\Theta = -\dot{\Theta}\, z \tag{58}$$

Hence, $\dot{\Theta} = -\dot{x}/z$ and Eq. (57) may be replaced by

$$\dot{x} = \alpha\left[-\frac{C_P\lambda z^2\Lambda/I_P}{(C_P/I_P)^2 + \lambda^2} - \frac{C_R\lambda z^2(\Lambda+\dot{\phi})/I_R}{(C_R/I_R)^2 + \lambda^2}\right]. \tag{59}$$

Equation (59) is one of the desired equations in normal form. The other two are

$$\dot{\Phi} = \Lambda + \mathcal{O}(\alpha) \tag{60a}$$

and

$$\dot{\Psi} = \lambda + \mathcal{O}(\alpha) \tag{60b}$$

Equations (59) and (60) are in the required normal form for the generalized method of averaging. Only Eq. (59) is expanded because it is the only one required to investigate nutational motion.

IV. FIRST ORDER SOLUTION FOR THE NUTATION ANGLE

The first-order solution for the nutation angle, in terms of x, can now be developed by applying the generalized method of averaging to Eq. (59). Initially, the following nondimensional ratios must be defined:

$$\mu_P = \frac{\delta_P \lambda}{\delta_P^2 + \lambda^2} \tag{60}$$

and

$$\mu_R = \frac{\delta_R \lambda}{\delta_R^2 + \lambda^2} . \tag{61}$$

Then Eq. (59) may be written as:

$$\dot{x} = \alpha[-\mu_P z^2 \Lambda - \mu_R z^2 (\Lambda + \dot{\phi})] \tag{62}$$

At this point one may apply the generalized method of averaging to Eq. (62). Referring to the procedure in Appendix B,

$$A_1(\bar{x}) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} X_1(x,\Phi,\Psi) d\bar{\Phi} d\bar{\Psi} \tag{63}$$

where

$$X_1(x,\Phi,\Psi) = -\mu_P z^2 \Lambda - \mu_R z^2 (\dot{\Lambda} + \phi)$$

Since X_1 does not contain periodic terms in Φ and Ψ, the averaged equation to first order in the new parameter variable α is

$$\dot{\bar{x}} = \alpha[-\mu_P \bar{z}^2 \bar{\Lambda} - \mu_R \bar{z}^2 (\bar{\Lambda} + \dot{\phi})] \tag{65}$$

where $\bar{z}$ and $\bar{\Lambda}$ are obtained by replacing x with $\bar{x}$ in z and Λ. Equation (65) may be solved exactly by separating variables and using

partial fractions. Befor proceeding, however, it is necessary to approximate $\bar{\Lambda}$ by using

$$\bar{\Lambda} = H\bar{x}\left(\frac{1}{J_p} - \frac{1}{I_t}\right) - h/J_p \tag{66}$$

and $h = J_R(\omega_3+\dot{\phi})$.

The approximation to ω_3 given by Eq. (17c) may be used to obtain

$$h = J_R[(H\bar{x}-h)/J_p + \dot{\phi}] + \mathcal{O}(\alpha) \; .$$

Hence,

$$h = H\bar{x}\left(\frac{J_R}{J_p+J_R}\right) + J_p\dot{\phi}\left(\frac{J_R}{J_p+J_R}\right) + \mathcal{O}(\alpha) \; . \tag{67}$$

The above expression may be substituted into Eq. (66) to obtain

$$\bar{\Lambda} = H\bar{x}\left(\frac{1}{J_p} - \frac{1}{I_t}\right) - \frac{H\bar{x}}{J_p}\left(\frac{J_R}{J_p+J_R}\right) - \dot{\phi}\left(\frac{J_R}{J_p+J_R}\right) \; . \tag{68}$$

In terms of H/I_t, $\dot{\phi}$, and inertia ratios, one has

$$\bar{\Lambda} = \frac{H\bar{x}}{I_t}\left(\frac{1-\sigma-\rho}{\sigma+\rho}\right) - \dot{\phi}\left(\frac{\sigma}{\sigma+\rho}\right), \tag{69}$$

where $\sigma = J_R/I_t$ and $\rho = J_p/I_t$.

When Eq. (69) is used in Eq. (65), it is found that

$$\dot{\bar{x}} = \bar{z}^2\left[\alpha(\mu_p+\mu_R)\left(\frac{\sigma+\rho-1}{\sigma+\rho}\right)\frac{H\bar{x}}{I_t} + \alpha\dot{\phi}\left(\frac{\mu_\rho\sigma}{\sigma+\rho} - \frac{\mu_R\rho}{\sigma+\rho}\right)\right] \; . \tag{70}$$

Equation (70) can be rewritten in the form,

$$\dot{\bar{x}} = (1-\bar{x}^2)(a+b\bar{x}) \; , \tag{71}$$

where

$$a = \alpha\dot{\phi}\ \left(\frac{\mu_P\sigma}{\sigma+\rho} - \frac{\mu_R\rho}{\sigma+\rho}\right) \tag{72}$$

and

$$b = \alpha(\mu_p+\mu_R)\left(\frac{\sigma+\rho-1}{\sigma+\rho}\right)\frac{H}{I_t}\ . \tag{73}$$

Now, by separating variables, using partial fractions and integrating Eq. (71) with $\bar{x} = x_o$ at $t = 0$ the following solution may be obtained.

$$t = -C_1\ \ell n[(1-\bar{x})/(1-x_o)] + C_2\ \ell n[(1+x)\bar{/}(1+x_o)]$$

$$+ C_3\ \ell n[(a+b\bar{x})/(a+bx_o)]\ , \tag{74}$$

where

$$C_1 = \frac{1}{2(a+b)}\ ,$$

$$C_2 = \frac{1}{2(a-b)}$$

and

$$C_3 = \frac{b}{a^2-b^2}\ .$$

V. STABILITY CONDITIONS

If the spacecraft's nominal attitude motion is asymptotically stable, the "averaged" nutation angle will decay to zero as time increases. Considering Eq. (71) for asymptotic stability, one sees that it is necessary that $\dot{\bar{x}}$ should be positive. This criteria may be expressed as

$$(1-\bar{x}^2)(a+b\bar{x}) > 0. \tag{75}$$

Since the term $(1-\bar{x}^2)$ is always positive for nutation angles greater than zero, it is necessary that

$$a + b\bar{x} > 0\ . \tag{76}$$

The substitution of the explicit expressions for a and b into Eq. (76) provides the inequality,

$$\bar{x} > \frac{\dot{\phi}(\mu_R\rho-\mu_P\sigma)}{H/I_t(\mu_P+\mu_R)(\sigma+\rho-1)}\ . \tag{77}$$

If the averaged value of x is used, then $\bar{H}_3 = H\bar{x}$, and the condition for stability is

$$\bar{H}_3/I_t(\mu_P+\mu_R)(\sigma+\rho-1) > \dot{\phi}(\mu_R\rho-\mu_P\sigma)\ . \tag{78}$$

If the stability of the nominal spacecraft motion in which the platform is inertially fixed and the rotor spins at a constant rate is considered, then, for small nutation angles

$$H_3/I_t = H/I_t = \sigma^*\Omega \ , \tag{79}$$

where $\sigma^* = \sigma + \alpha$ and $\dot{\phi} = \Omega$ is a constant, then Eq. (78) may be put into the form,

$$\mu_P(\sigma+\rho-1 + \sigma/\sigma^*) + \mu_R(\sigma+\rho-1-\rho/\sigma^*) > 0 \ . \tag{80}$$

For a specific set of inertia parameters, Eq. (80) can be used to define regions of stability in the $\delta_P - \delta_R$ plane.

VI. RESULTS

Small Nutation Angles

If condition (80) is examined in the limiting case where $\delta_R \to 0$, the requirement for stability is found to be

$$\sigma + \rho + \sigma/\sigma^* > 1 \ , \tag{81}$$

which is always true in the small "α" approximation. This case corresponds to energy dissipation on the platform only and the motion should be stable.[2]

In the other limiting case where $\delta_P \to 0$, the requirement for stability is

$$\sigma + \rho > 1 + \rho/\sigma^*. \tag{82}$$

A careful examination of condition (82) reveals that a value of $\sigma > 1$ would insure stability of nutational motion in this case. This condition corresponds to an oblate spacecraft (i.e., one that spins about its axis of maximum moment of inertia) which, according to previous results, also exhibits stable behavior.[1]

When energy dissipation is present on both the rotor and platform, condition (80) can be examined for specific inertia parameters. Figures 3 through 7 are stability diagrams for δ_P/Ω vs. δ_R/Ω for different spacecraft parameters. Fig. 3 shows a critical case where σ and ρ both equal 0.5.

Some qualitative results can be obtained by examination of Figs. 4 through 7. These figures indicate that if the rotor inertia parameter

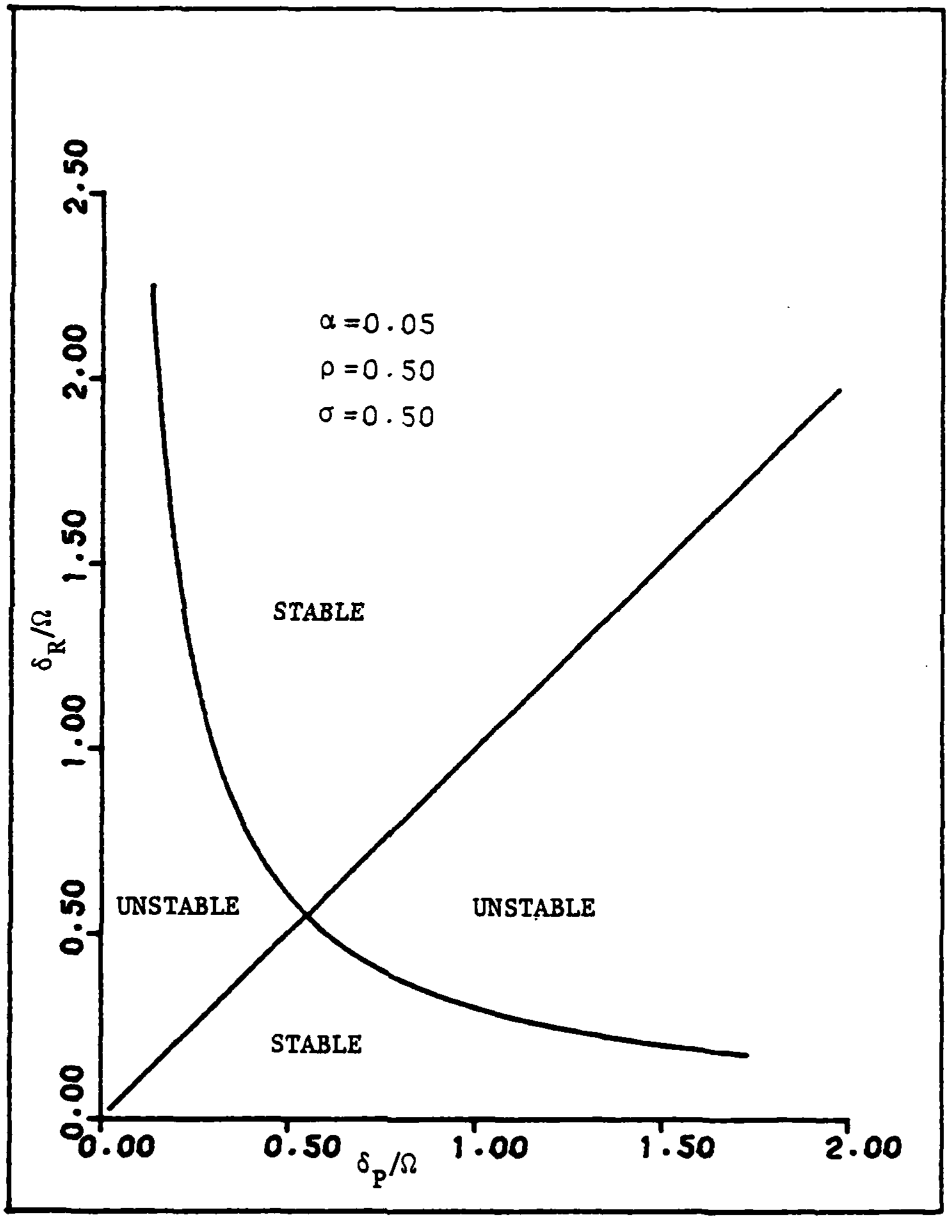

Figure 3. Stability plot in the damping parameter plane for α=0.05, ρ=0.5, σ=0.5.

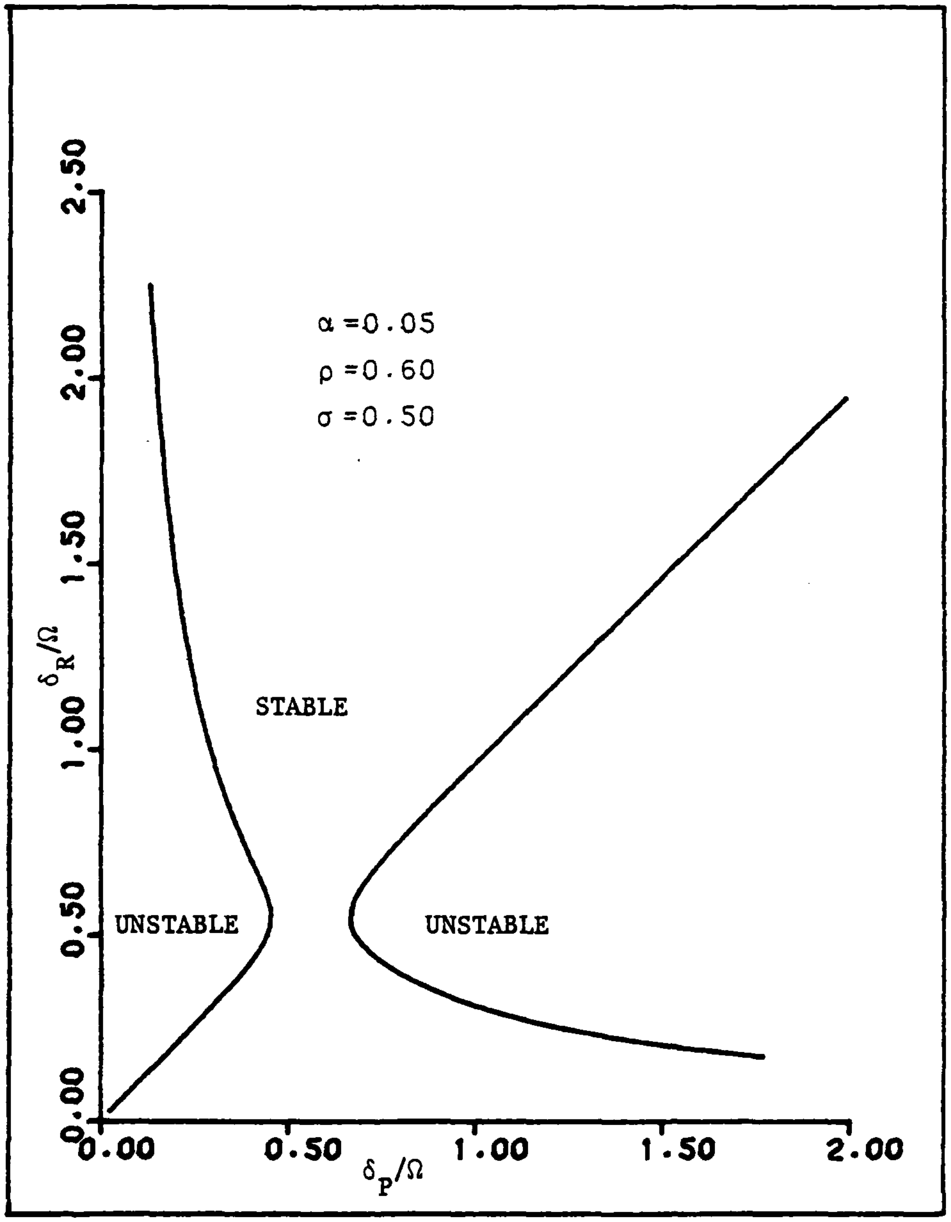

Figure 4. Stability plot in the damping parameter plane for α=0.05, ρ=0.6, σ=0.5.

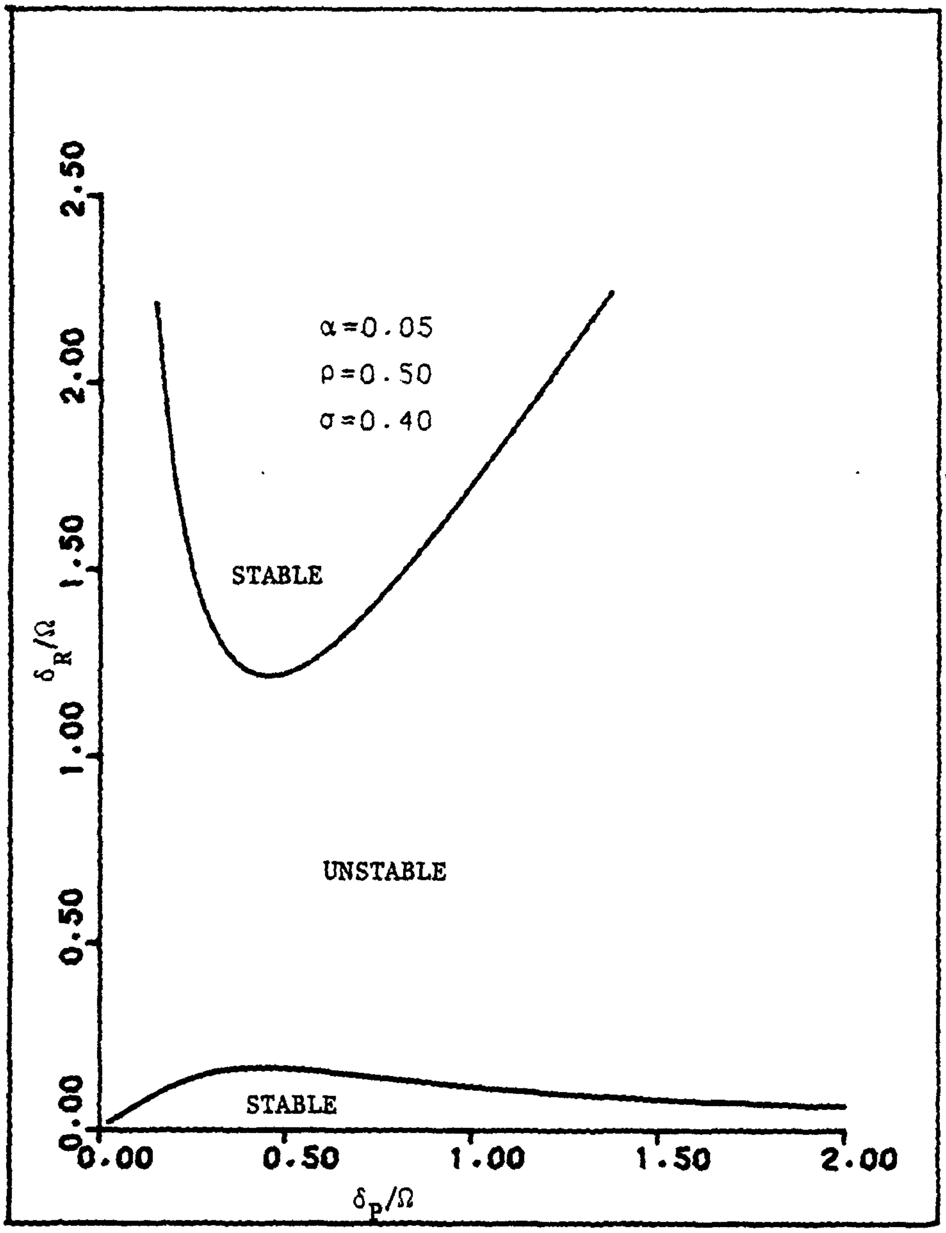

Figure 5. Stability plot in the damping parameter plane for $\alpha=0.05$, $\rho=0.5$, $\sigma=0.4$.

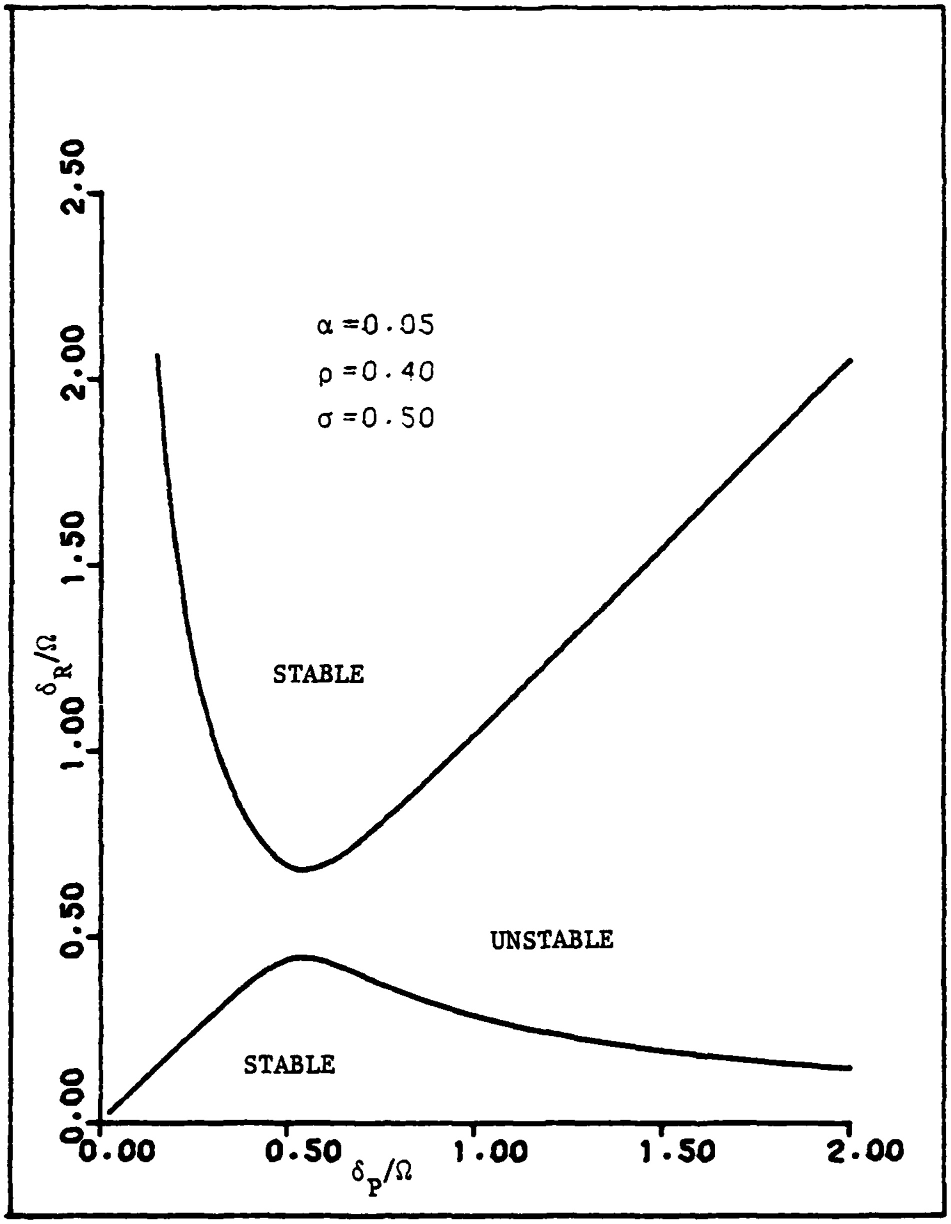

Figure 6. Stability plot in the damping parameter plane for $\alpha=0.05$, $\rho=0.4$, $\sigma=0.5$.

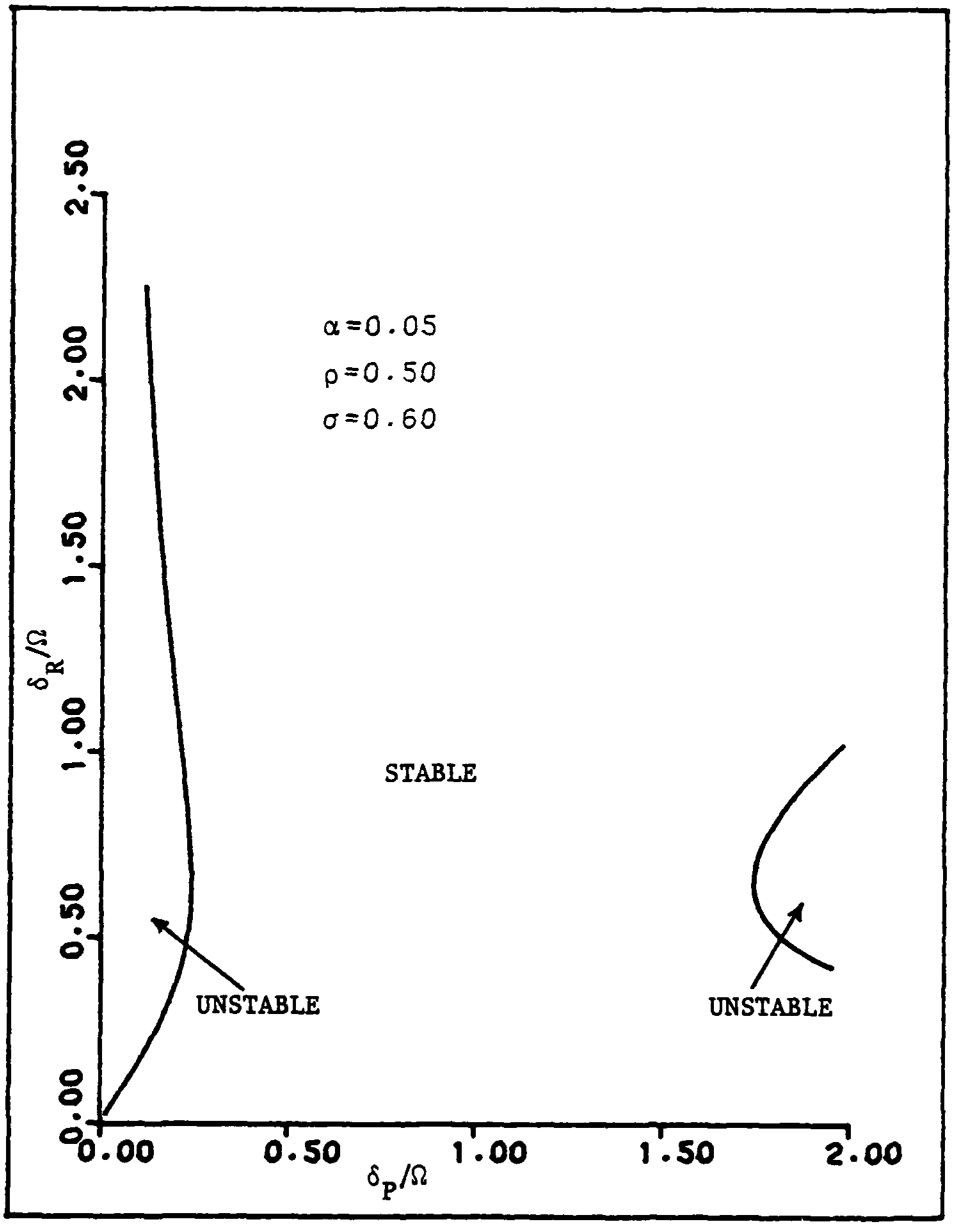

Figure 7. Stability plot in the damping parameter plane for α=0.05, ρ=0.5, σ=0.6.

is larger than the platform inertia parameter, then more of the area in the region considered predicts stable motion, while if the platform inertia parameter is larger, then more of the region predicts unstable motion. In Figs. 4 and 5, the platform ratio is the larger and most of the area predicts unstable motion, especially in Fig. 5 where the rotor parameter is less than the critical value of 0.5. In Figs. 6 and 7, the rotor ratio is the larger of the two and most of the area predicts stable motion, especially in Fig. 7 where the rotor parameter is larger than 0.5. It should be noted here that Figs. 3 through 7 are very similar to the stability diagrams found by using a linear analysis.[4] However, the results of this analysis contain the effects of inertia parameters for both the rotor and platform, while the linear analysis results depend only on the rotor inertia parameter.

Nonlinear Stability

Another result of the generalized method of averaging is that an analytical approximation to the nutational motion can be found from Eq. (74). Figures 8 through 11 show how the approximate solution of Eq. (74) compares to the "exact" numerical solution of the full nonlinear equations (see Appendix A). The full nonlinear equations were solved using a fourth-order Runge-Kutta method.

To compare the results of the approximate solution to the Runge-Kutta solution, two cases are considered; one with the platform inertia parameter larger than the rotor inertia parameter and one with the rotor inertia parameter the larger. The predicted subcases, one stable and one unstable, of each case are considered. The nonlinear equations are

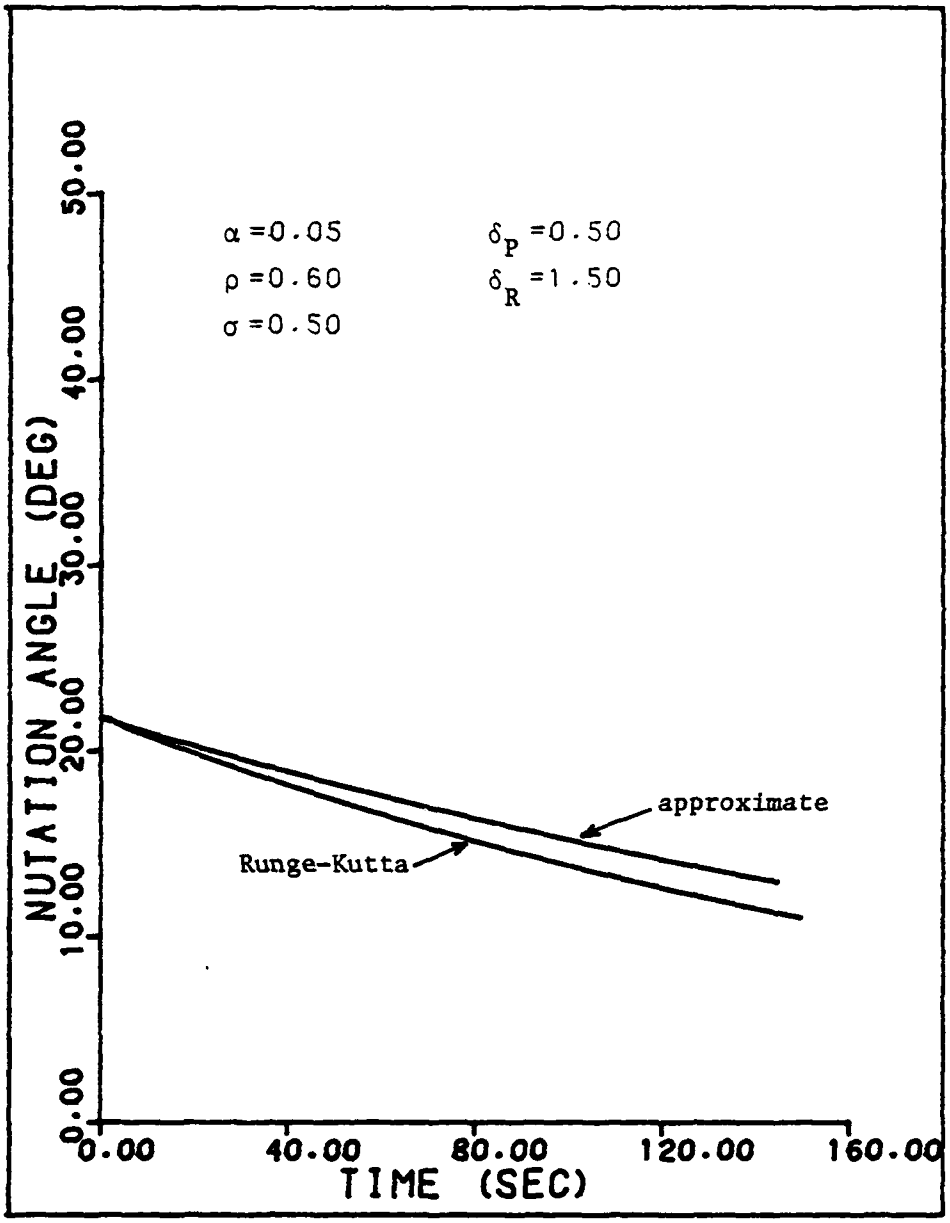

Figure 8. Simulation run with the parameters: α=0.05, ρ=0.6, σ=0.5, δ_P=0.5, δ_R=1.5.

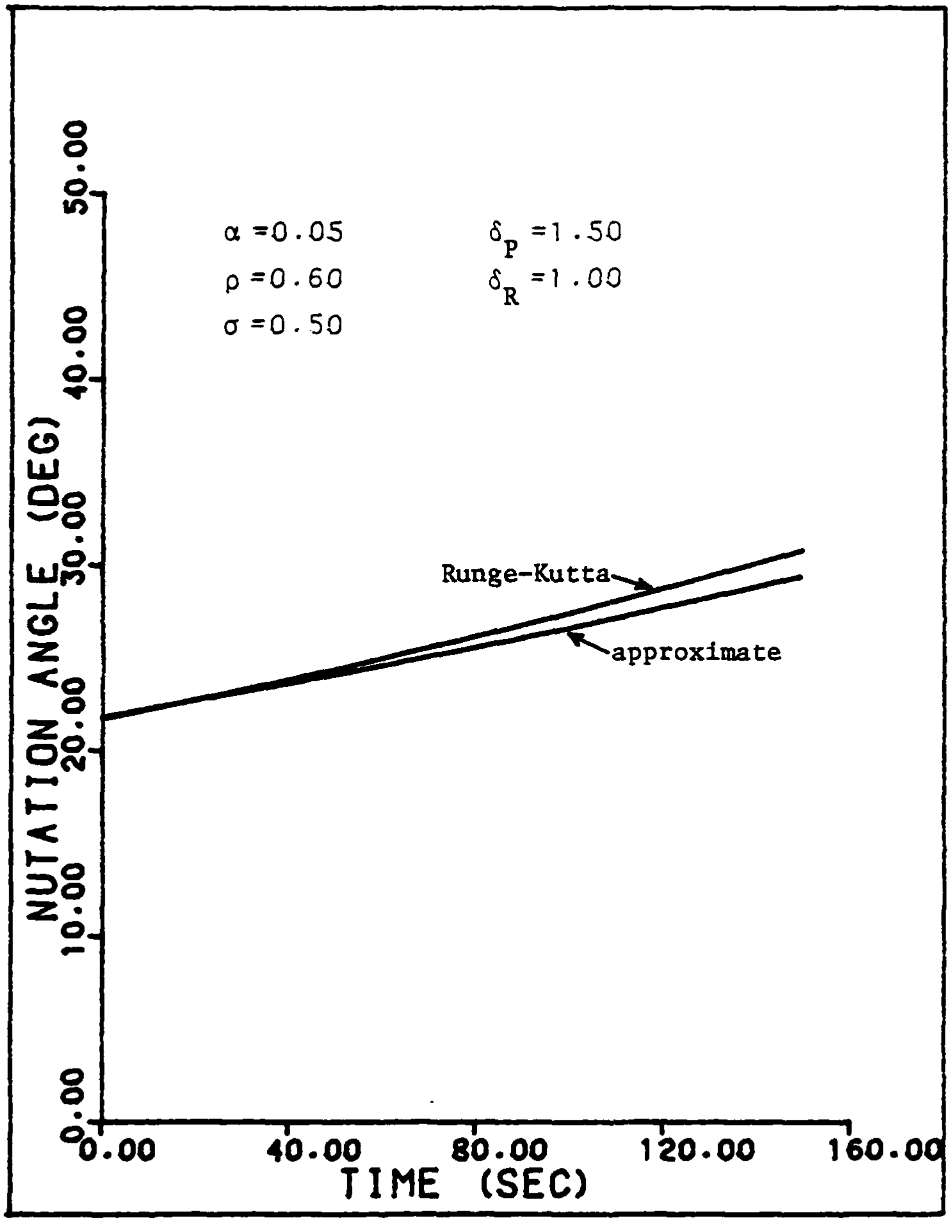

Figure 9. Simulation run with the parameters: α=0.05, ρ=0.6 σ=0.5, δ_P=1.5, δ_R=1.0.

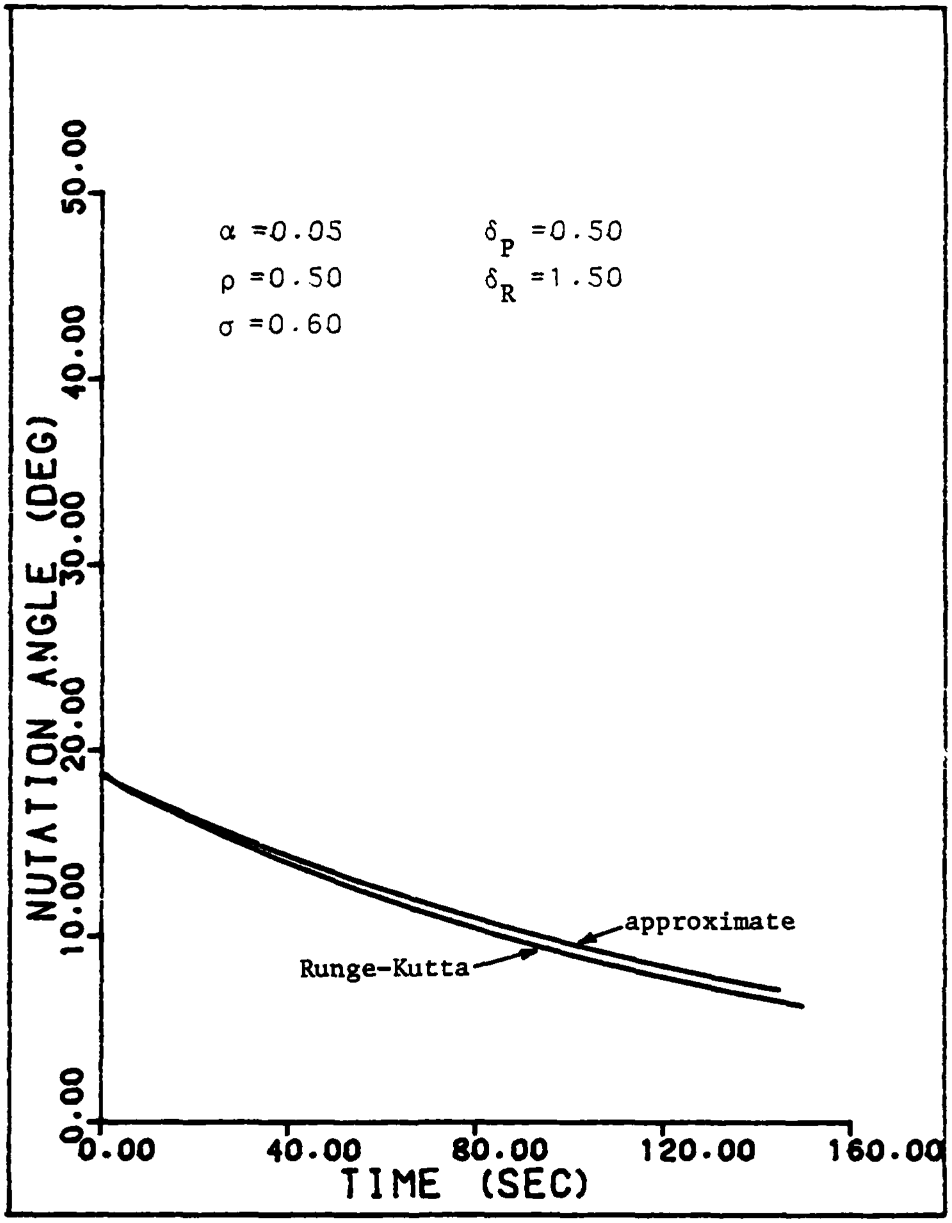

Figure 10. Simulation run with the parameters: α=0.05, ρ=0.5 σ=0.6, δ_P=0.5, δ_R=1.5.

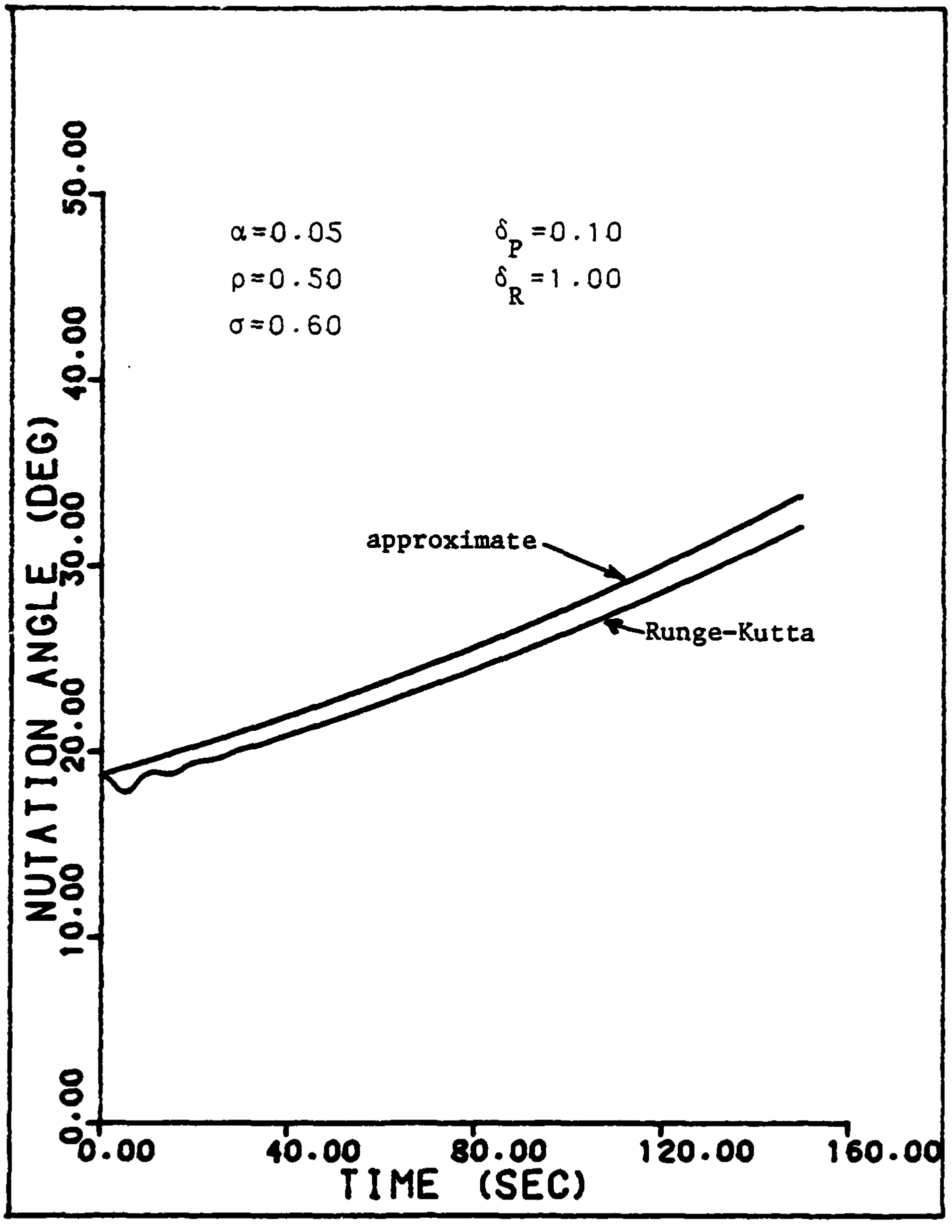

Figure 11. Simulation run with the parameters: α=0.05, ρ=0.5, σ=0.6, δ_P=0.1, δ_R=1.0.

integrated with $\ddot{\phi} = 0$ and initial conditions of $\omega_{10} = 0.2$ rad/sec, $\Omega = 1.0$ rad/sec and all others zero.

Figures 8 and 9 show the results obtained when the platform inertia parameter is larger than the rotor inertia parameter and the approximate results are in good agreement with the numerical results. Figures 10 and 11 show the results for the case when the rotor parameter is larger than the platform inertia ratio and again the results agree well, especially in the stable case of Fig. 10. In Fig. 11, the transient nonlinearities seem to increase the difference between the approximate and numerical results. In all cases, the difference between the two solutions appears to be a linear function of time.

Figures 12 and 13 show the results obtained for cases of no damping on the rotor (Fig. 12) and no damping on the platform (Fig. 13), respectively. These figures show the effects of neglecting the transient terms in Eqs. (36) and (50). When there is no damping, the oscillatory terms never disappear. However, that these terms do eventually vanish if only a small amount of damping is present is illustrated by Fig. 14. the transient terms die out very rapidly when significantly large damping constants are used

Figures 15 and 16 demonstrate the results of "fixing" the damper on the rotor and platform, respectively. These simulation results were obtained by setting the damping ratio to zero, while at the same time setting $\dot{\omega}_{R1} = \dot{\omega}_{R2} = \dot{\omega}_{R3} = 0$, for Fig. 15, and $\dot{\omega}_{P1} = \dot{\omega}_{P2} = \dot{\omega}_{P3} = 0$, for Fig. 16, in the full set of equations. Figures 15 and 16 are analogous to Figs. 12 and 13, respectively. Again, the analytical and numerical results compare quite well, especially for the stable case.

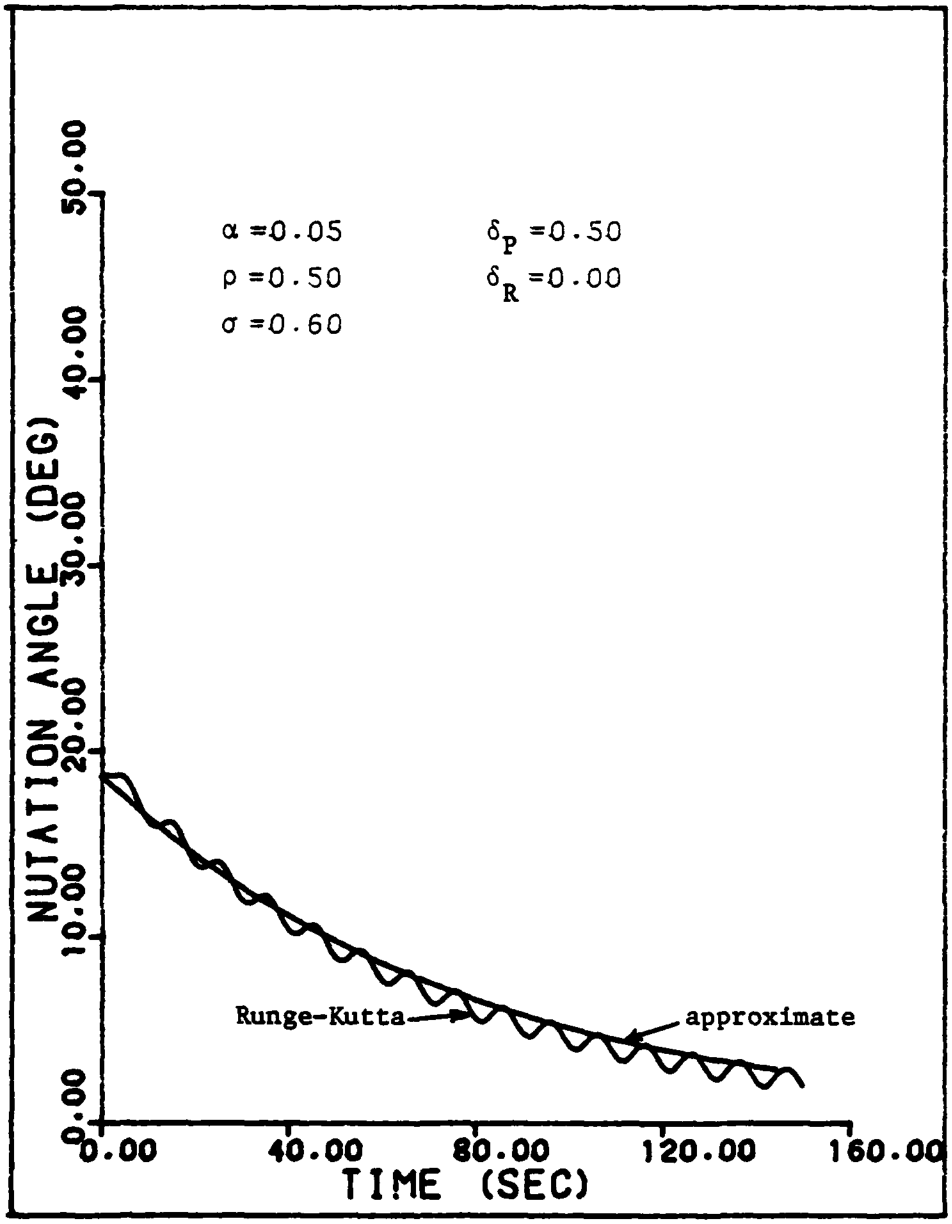

Figure 12. Simulation run with the parameters: α=0.05, ρ=0.5 σ=0.6, δ_P=0.5, δ_R=0.0.

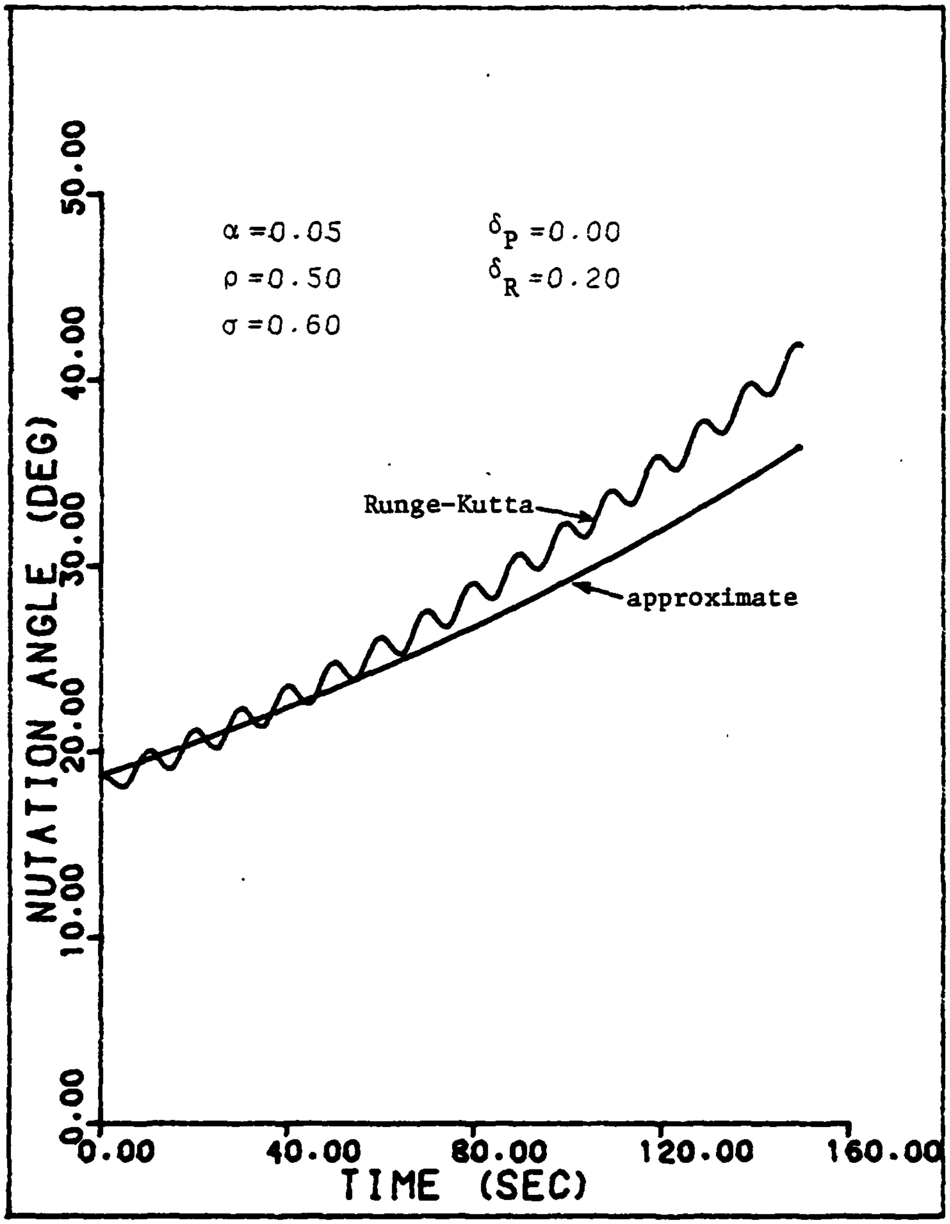

Figure 13. Simulation run with the parameters: $\alpha=0.05$, $\rho=0.5$ $\sigma=0.6$, $\delta_P=0.0$, $\delta_R=0.2$.

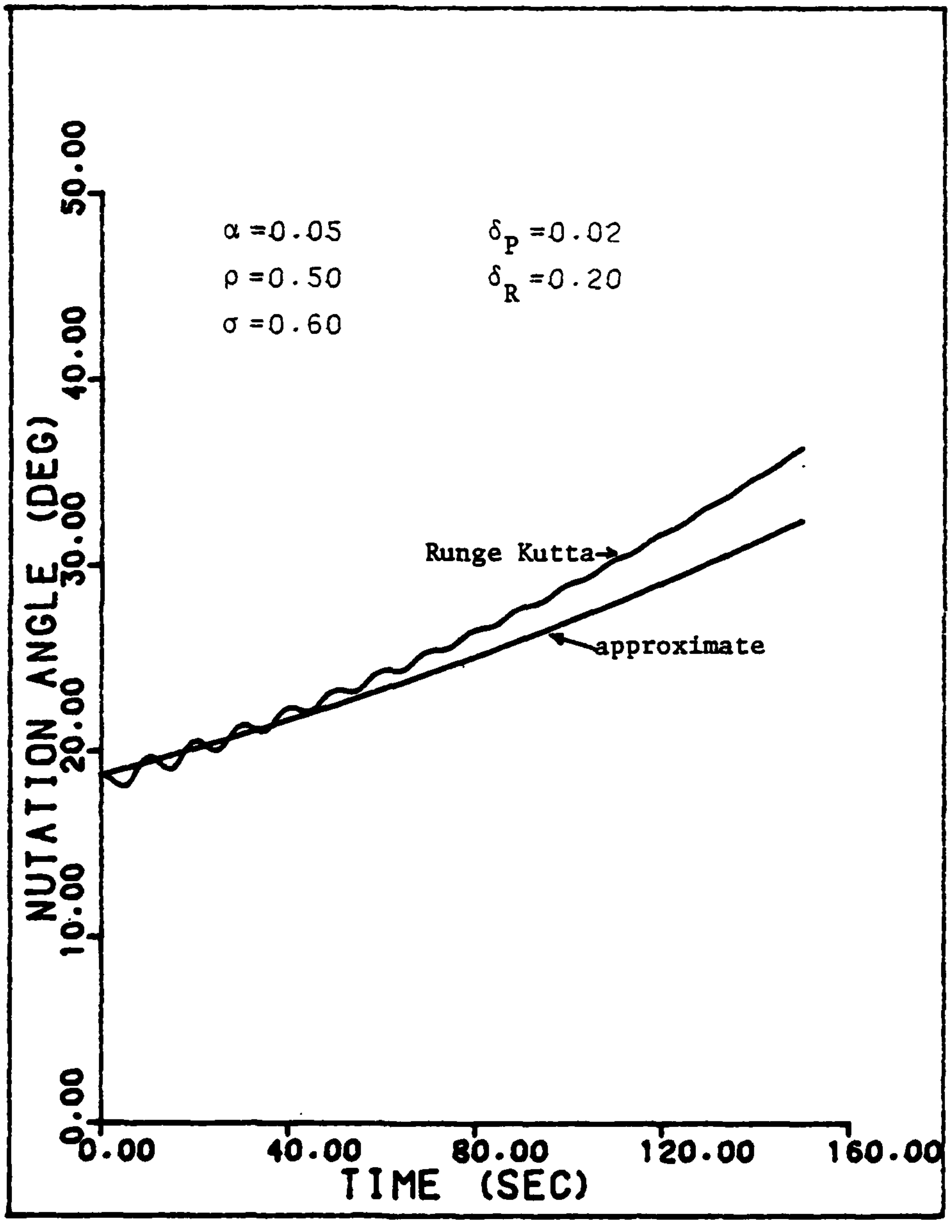

Figure 14. Simulation run with the parameters: α=0.05, ρ=0.5 σ=0.6, δ_P=0.02, δ_R=0.2

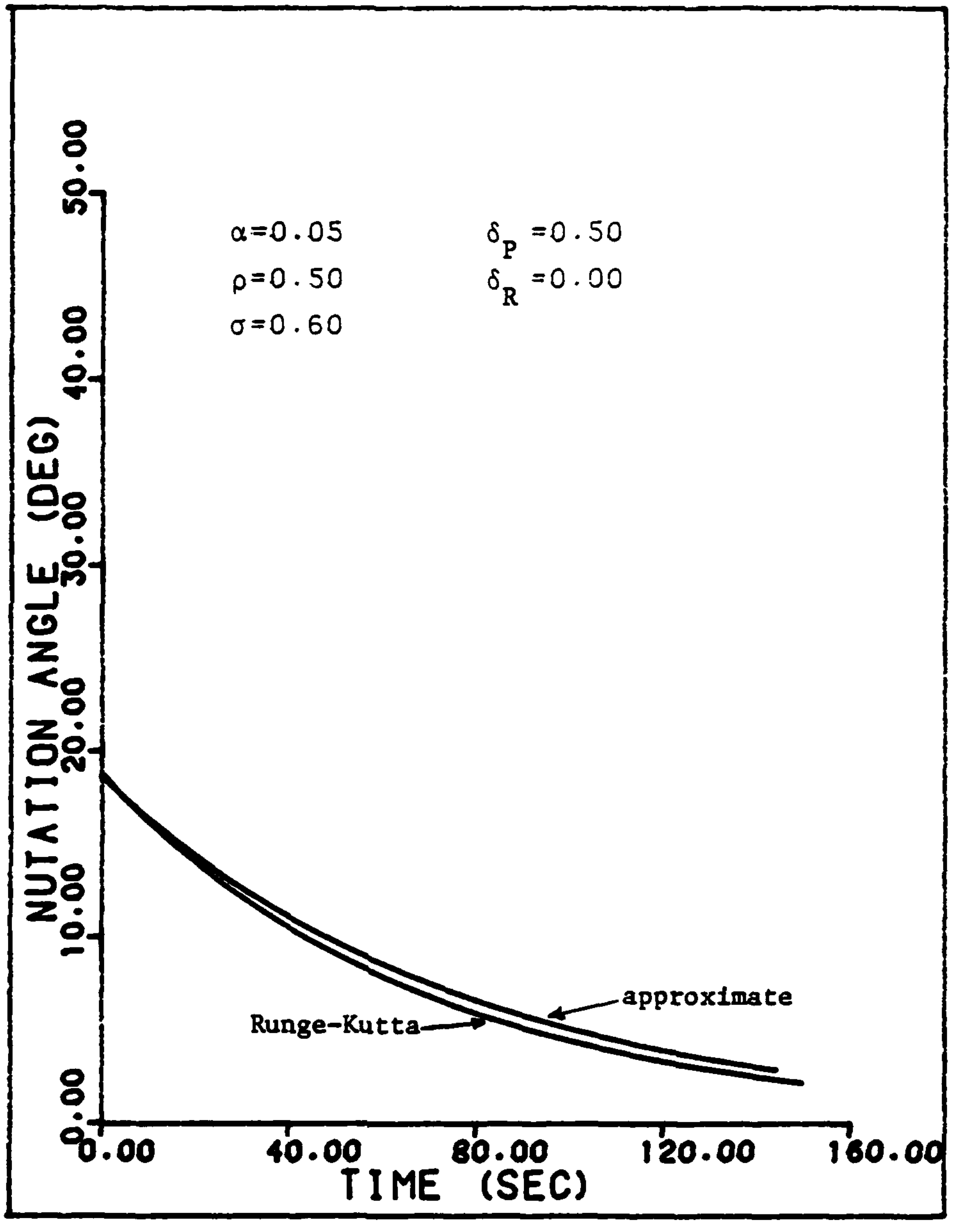

Figure 15. Simulation with the parameters: α=0.05, ρ=0.5, σ=0.6, δ_P=0.5, δ_R=0.0, and the rotor damper fixed.

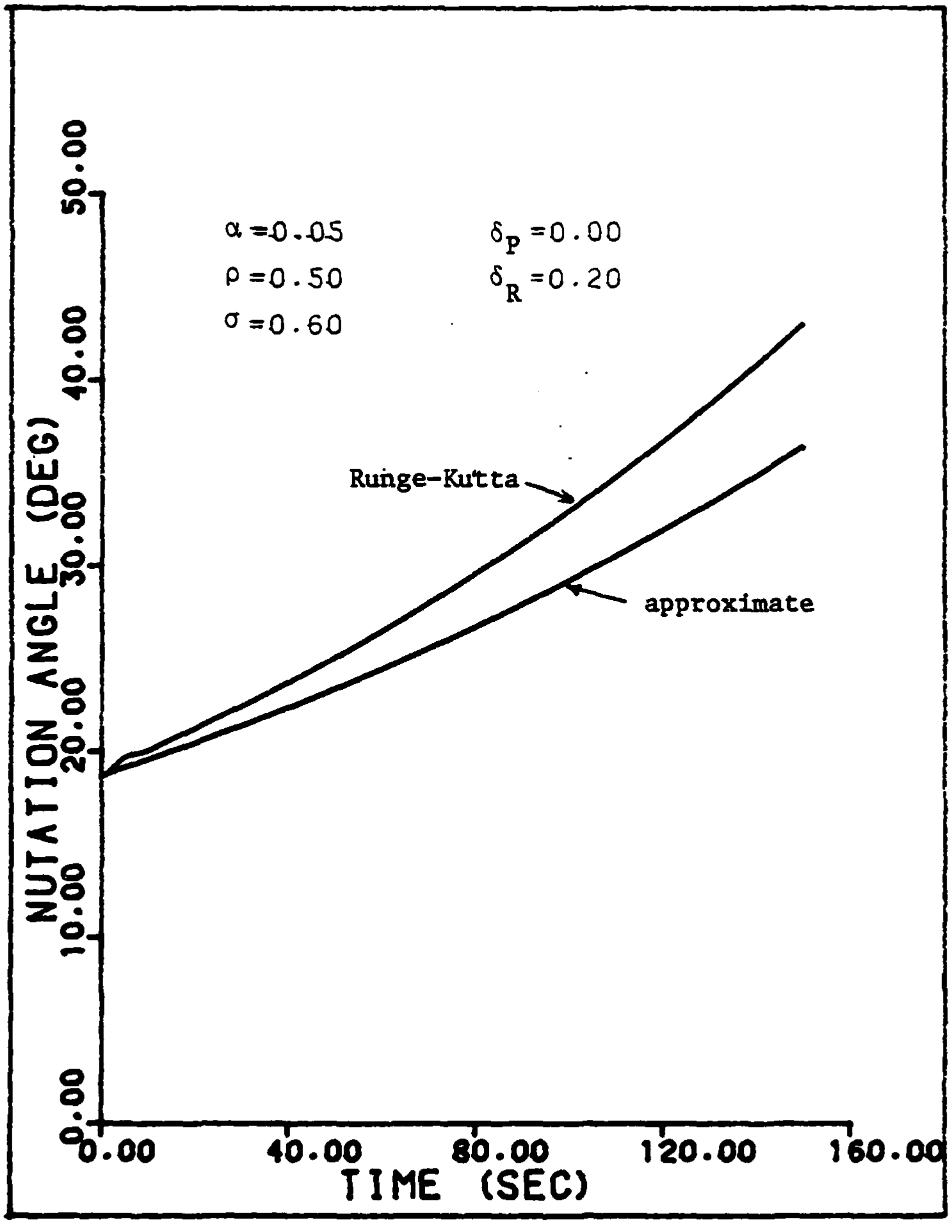

Figure 16. Simulation run with the parameters: α=0.05, ρ=0.5, σ=0.6, δ_P=0.0, δ_R=0.2, and the platform damper fixed.

VII. CONCLUSIONS

The attitude motion of a model of a symmetric dual-spin spacecraft containing spherical dampers on both its rotor and platform has been investigated. A perturbation method which treats the effects of the motion of the dampers as perturbing torques has been used in conjunction with the generalized method of averaging. A stability analysis was conducted for the constant speed rotor case and produced results that agree well with the linear stability analysis described in Ref. 4. However, the analysis of this thesis is more general in that more information regarding the inertia parameters is obtained than is possible using a conventional linear analysis. The generalized method of averaging also provides an approximate solution for the nutational motion which is not restricted to small angles. This solution agrees well with results obtained by numerically integrating the full set of nonlinear equations.

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APPENDICES

APPENDIX A

DERIVATION OF THE SPACECRAFT EQUATIONS OF MOTION

APPENDIX A

The equations of motion of the spacecraft are derived by applying Euler's moment equation, to the spacecraft as a whole, then to the rotor and platform dampers individually, and then to the rotor subsystem.

The angular momentum of the entire system about its center of mass is

$$H_1 = I_t\omega_1 + I_P(\omega_{P1}+\omega_1) + I_R(\omega_{R1}+\omega_1), \tag{A-1a}$$

$$H_2 = I_t\omega_2 + I_P(\omega_{P2}+\omega_2) + I_R(\omega_{R2}+\omega_2) \tag{A-1b}$$

and

$$H_3 = J_P\omega_3 + J_R(\omega_3+\dot{\phi}) + I_P(\omega_{P3}+\omega_3) + I_R(\omega_{R3}+\omega_3+\dot{\phi}) \tag{A-1c}$$

Let $I_t^* = I_t + I_P + I_R$ and $I_s^* = J_P + J_R + I_P + I_R$

Then, in terms of I_s^* and I_t^*, one has

$$H_1 = I_t^*\omega_1 + I_P\omega_{P1} + I_R\omega_{R1} \tag{A-2a}$$

$$H_2 = I_t^*\omega_2 + I_P\omega_{P2} + I_R\omega_{R2} \tag{A-2b}$$

and

$$H_3 = I_s^*\omega_3 + I_P\omega_{P3} + I_R\omega_{R3} + (J_R+I_R)\dot{\phi} \tag{A-2c}$$

Now, by applying Euler's moment equation to Eqs. (A-2), one finds that

$$\underline{M} = \dot{\underline{H}} + \underline{\omega}\times\underline{H} \tag{A-3}$$

Since the spacecraft is not affected by external torques

$$\overset{\circ}{\underline{H}} + \underline{\omega} x \underline{H} = \underline{0} \ . \tag{A-4}$$

The component form of Eq. (A-4) is

$$I_t^* \dot{\omega}_1 + I_P \dot{\omega}_{P1} + I_R \dot{\omega}_{R1} + \omega_2 H_3 - \omega_3 H_2 = 0 \tag{A-5a}$$

$$I_t^* \dot{\omega}_2 + I_P \dot{\omega}_{P2} + I_R \dot{\omega}_{R2} + \omega_3 H_2 - \omega_1 H_3 = 0 \tag{A-5b}$$

$$I_s^* \dot{\omega}_3 + I_P \dot{\omega}_{P3} + I_R \dot{\omega}_{R3} + (J_R + I_R)\ddot{\phi} + \omega_1 H_2 - \omega_2 H_1 = 0 \ . \tag{A-5c}$$

By substituting for H_1, H_2, and H_3 and simplifying Eq. (A-5) one gets

$$I_t^* \dot{\omega}_1 + I_P \dot{\omega}_{P1} + I_R \dot{\omega}_{R1} + (J_R + I_R)\dot{\phi}\omega_2 + (I_s^* - I_t^*)\omega_2\omega_3 + I_P(\omega_{P3}\omega_2 - \omega_{P2}\omega_3) + I_R(\omega_{R3}\omega_2 - \omega_{R2}\omega_3) = 0, \tag{A-6a}$$

$$I_t^* \dot{\omega}_2 + I_P \dot{\omega}_{P2} + I_R \dot{\omega}_{R2} - (J_R + I_R)\dot{\phi}\omega_1 + (I_t^* - I_s^*)\omega_1\omega_3 + I_P(\omega_{P1}\omega_3 - \omega_{P3}\omega_1) + I_R(\omega_{R1}\omega_3 - \omega_{R3}\omega_1) = 0 \tag{A-6b}$$

and

$$I_s^* \dot{\omega}_3 + I_P \dot{\omega}_{P3} + I_R \dot{\omega}_{R3} + (J_R + I_R)\ \ddot{\phi} + I_P(\omega_{P2}\omega_1 - \omega_{P1}\omega_2) + I_R(\omega_{R2}\omega_1 - \omega_{R1}\omega_2) = 0 \ . \tag{A-6c}$$

The platform-fixed components of angular momentum for the platform dampers can be written as

$$h_{P1} = I_P(\omega_{P1} + \omega_1), \tag{A-7a}$$

$$h_{P2} = I_P(\omega_{p2} + \omega_2) \tag{A-7b}$$

and

$$h_{P3} = I_P(\omega_{P3} + \omega_3). \tag{A-7c}$$

Application of Euler's moment equation to the platform damper yields

$$\underline{M} = \overset{\circ}{\underline{h}} + \underline{\omega} x \underline{h}_P = \underline{T}_P \tag{A-8}$$

where T_{-P} is a 3x1 matrix of the components of the torque applied to the platform dampers. Equations (A-8) may be written in component form, viz,

$$I_P(\dot{\omega}_{P1}+\dot{\omega}_1) + \omega_2 h_{P3} - \omega_3 h_{P2} = T_{P1} \tag{A-9a}$$

$$I_P(\dot{\omega}_{P2}+\dot{\omega}_2) + \omega_3 h_{P1} - \omega_1 h_{P3} = T_{P2} \tag{A-9c}$$

and

$$I_P(\dot{\omega}_{P3}+\dot{\omega}_3) + \omega_1 h_{P2} - \omega_2 h_{P1} = T_{P3} \tag{A-9c}$$

One may replace $T_{Pi} = -C_P\omega_{Pi}$ for i = 1,2,3 and substitute for h_{Pi} for i = 1,2,3 to get the equations,

$$I_P(\dot{\omega}_{P1}+\dot{\omega}_1) + I_P(\omega_{P3}\omega_2-\omega_{P2}\omega_3) + C_P\omega_{P1} = 0, \tag{A-10a}$$

$$I_P(\dot{\omega}_{P2}+\dot{\omega}_2) + I_P(\omega_{P1}\omega_3-\omega_{P3}\omega_1) + C_P\omega_{P2} = 0 \tag{A-10b}$$

and

$$I_P(\dot{\omega}_{P3}+\dot{\omega}_3) + I_P(\omega_{P2}\omega_1-\omega_{P1}\omega_2) + C_P\omega_{P3} = 0 \tag{A-10c}$$

Similarly, the angular momentum for the rotor dampers can be written as

$$h_{R1} = I_R(\omega_{R1}+\omega_1), \tag{A-11a}$$

$$h_{R2} = I_R(\omega_{R2}+\omega_2) \tag{A-11b}$$

and

$$h_{R3} = I_R(\omega_{R3}+ \omega_3+\dot{\phi}) \tag{A-11c}$$

and the application of Euler's moment equation to the rotor dampers provides

$$I_R(\dot{\omega}_{R1}+\dot{\omega}_1) + \omega_2 h_{R3} - \omega_3 h_{R2} = T_{R1}, \tag{A-12a}$$

$$I_R(\dot{\omega}_{R2}+\dot{\omega}_2) + \omega_3 h_{R1} - \omega_1 h_{R3} = T_{R2} \tag{A-12b}$$

and

$$I_R(\dot{\omega}_{R3}+ \dot{\omega}_3+ \ddot{\phi}) + \omega_1 h_{R2} - \omega_2 h_{R1} = T_{R3}. \tag{A-12c}$$

By using $T_{Ri} = C_R \omega_{Ri}$, $i = 1,2,3$, and substituting h_{Ri} for $i = 1,2,3$, one may put Eqs (A-12) in the form,

$$I_R(\dot{\omega}_{R1}+\dot{\omega}_1) + I_R(\omega_{R3}\omega_2-\omega_{R2}\omega_3) + I_R\dot{\phi}\omega_2 + C_R\omega_{R1} = 0, \tag{A-13a}$$

$$I_R(\dot{\omega}_{R2}+\dot{\omega}_2) + I_R(\omega_{R1}\omega_3-\omega_{R3}\omega_1) - I_R\dot{\phi}\omega_1 + C_R\omega_{R2} = 0 \tag{A-13b}$$

and

$$I_R(\dot{\omega}_{R3}+\dot{\omega}_3+\ddot{\phi}) + I_R(\omega_{R2}\omega_1-\omega_{R1}\omega_2) + C_R\omega_{R3} = 0. \tag{A-13c}$$

By considering the rotor as a system, one may derive three equations for the rotation motion of the rotor; however, only the equation corresponding to the rotation about $\hat{\underline{e}}_3$ is needed. This equation corresponds to the tenth degree of freedom of the system.

If $\{R\}$ is used to denote the 3x1 matrix of platform-fixed components of the angular momentum of the rotor and J_t as its transverse moment of inertia, the following equations may be written:

$$R_1 = J_t\omega_1 + I_P(\omega_{P1}+\omega_1) + I_R(\omega_{R1}+\omega_1), \tag{A-14a}$$

$$R_2 = J_t\omega_2 + I_P(\omega_{P2}+\omega_2) + I_R(\omega_{R2}+\omega_2) \tag{A-14b}$$

and

$$R_3 = J_R(\omega_3+\dot{\phi}) + I_P(\omega_{P3}+\omega_3) + I_R(\omega_{R3}+\omega_3+\dot{\phi}). \tag{A-14c}$$

Euler's moment equation may be applied to Eqs. (A-14) to get

$$\underset{\sim}{M} = \dot{\underset{\sim}{R}} + \underset{\sim}{\omega} x \underset{\sim}{R} \ . \tag{A-15}$$

Considering only the third component of Eq. (A-15), one gets

$$(J_R+I_R)(\dot{\omega}_3+\ddot{\phi}) + I_R\dot{\omega}_{R3} + I_R(\omega_{R2}\omega_1-\omega_{R1}\omega_2) = 0 \ . \tag{A-16}$$

APPENDIX B

GENERALIZED METHOD OF AVERAGING

APPENDIX B

The basic idea behind the method of averaging is to derive an approximate solution of the nonautonomous system by considering in its place an associated average system, which is autonomous.[6] For the generalized method of averaging to be used, the differential equations to be solved must first be cast in "normal form."[6] The algorithm described below applies to the system of differential equations in this thesis which may be expressed as

$$\dot{x} = \alpha X_1(x,\Phi,\Psi), \tag{B-1a}$$

$$\dot{\Phi} = \Lambda(x) + \alpha Y_1(x,\Phi,\Psi) \tag{B-1b}$$

and

$$\dot{\Psi} = \lambda(x) + \alpha Z_1(x,\Phi,\Psi), \tag{B-1c}$$

where X_1, Y_1, and Z_1 are periodic with a period of 2π in the angles Φ and Ψ, and α is a small constant. An approximate solution to the system which is valid through first order in α may be obtained by solving the "averaged" equations:

$$\dot{\bar{x}} = \alpha\ A_1(\bar{x}), \tag{B-2a}$$

$$\dot{\bar{\Phi}} = \Lambda(\bar{x}) + \alpha\ B_1(\bar{x}) \tag{B-2b}$$

and

$$\dot{\bar{\Psi}} = \lambda(\bar{x}) + \alpha\ C_1(\bar{x}). \tag{B-2c}$$

Here,

$$A_1 = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} X_1(x,\Phi,\Psi)d\bar{\Phi}d\bar{\Psi} \quad , \tag{B-3a}$$

$$B_1 = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} Y_1(x,\Phi,\Psi)d\bar{\Phi}d\bar{\Psi} \quad , \tag{B-3b}$$

$$C_1 = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} Z_1(x,\Phi,\Psi)d\bar{\Phi}d\bar{\Psi} \quad , \tag{B-3c}$$

$$\Lambda(\bar{x}) = \Lambda(x) \tag{B-4a}$$

and

$$\lambda(\bar{x}) = \lambda(x). \tag{B-4b}$$

The transformation from the averaged system is

$$x = \bar{x} + \alpha\ u_1(\bar{x},\bar{\Phi},\bar{\Psi}), \tag{B-5a}$$

$$\Phi = \bar{\Phi} + \alpha\ v_1(\bar{x},\bar{\Phi},\bar{\Psi}) \tag{B-5b}$$

and

$$\Psi = \bar{\Psi} + \alpha\ w_1(\bar{x},\bar{\Phi},\bar{\Psi}), \tag{B-5c}$$

where

$$u_1 = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} (X_1 - A_1)d\bar{\Phi}d\bar{\Psi} \quad , \tag{B-6a}$$

$$v_1 = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} (Y_1 - B_1)d\bar{\Phi}d\bar{\Psi} \tag{B-6b}$$

and

$$w_1 = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} (Z_1 - C_1)d\bar{\Phi}d\bar{\Psi} \quad . \tag{B-6c}$$

In the present application, the expression for x is the one of dominant interest.